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1M3N20200 (Pages : 2) Reg. No:…………………..

Name: ……………………

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Degree Examination, November 2020

MMT3C11 – Multivariable Calculus & Geometry

(2019 Admission onwards)

Time: 3 hours Max. Weightage : 30

PARTA

Answer ALL questions. Each question carries 1 weightage.

1. Let A $\in L(R^n, R^m)$. Then prove that A is a uniformly continuous mapping of $R^n\ to\ R^m$.
2. Let $f: R^2 \to R^3$ be given by $f(x,y,z) = x^2 + y^2 + z^2$. Find the directional derivative of f at (1,1,1) in the direction of the vector $\left(\frac{4}{5}, 0, \frac{3}{5}\right)$.
3. State the Inverse Function Theorem.
4. Find the parametrised equation of the level curve $y^2 - x^2 = 1$.
5. Compute the curvature of the curve $\gamma(t) = \left(\frac{4}{5}cost, 1 - sint, -\frac{3}{5}cost\right)$.
6. Check whether $\sigma(u,v) = (u,v,uv)\ u,v \in R$ is a regular surface patch.
7. Show that first fundamental form for the plane $\sigma(u,v) = a + u\boldsymbol{p} + v\boldsymbol{q}$ in R^3 is $du^2 + dv^2$.
8. Define Weingarten map.

(8× 1 = 8 *weightage*)

PART B

Answer any two questions from each unit. Each question carries 2 weightage.

Unit I

9. Prove that a linear operator A on a finite dimensional space X is one to one if and only if the range of A is all of X.
10. Prove that $L(R^n, R^m)$ is a metric space.
11. Prove that if [A] and [B] are $n \times n$ matrices, then $\det[B]\,[A]) = \det[B]\det[A]$.

Unit II

12. Prove that the total signed curvature of a closed plane curve is an integer multiple of 2π.
13. Let γ be a unit speed curve in R^3 with constant curvature and zero torsion. Then, prove that γ is a parametrization of (part) of a circle.
14. If $f: S \to \tilde{S}$ is a smooth map between surfaces and $p \in S$, then prove that the derivative Map $D_P f: T_P S \to T_P \tilde{S}$ is a linear map.

Unit III

15. Compute the second fundamental form of the elliptic paraboloid

$\sigma(u,v) = (u, v, u^2 + v^2)$.

16. Calculate the Gauss map of the paraboloid S with equation $z = x^2 + y^2$.

17. Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms

$Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ respectively. Prove that the mean curvature is $\frac{LG-2MF+NE}{2(EG-F^2)}$.

$(6 \times 2 = 12$ *weightage*$)$

PART C

Answer any Two questions. Each question carries 5weightage.

18. a)Suppose f maps an open set $E \subset R^n$ into R^m, and f is differentiable at a point $x \in E$. Then prove that the partial derivatives $D_j f_i(x) (1 \le j \le n, 1 \le i \le m)$ exist at all points of E.

b) If $f(0,0) = 0$ and $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$,then prove that the function f is not differentiable in R^2 even though all the partial derivatives of f exist at all point of R^2.

19. a) State and prove the Contraction principle.

b) Give an example of a contraction on (0,1) having no fixed point.Does this contradict the contraction principle?

20. a)Prove that any reparametrisation of a regular curve is regular.

b) Prove that a parametrized curve has a unit speed reparametrization if and only if it regular.

21 a)Let $\sigma: U \to R^3$ be a surface patch. Let $(u_0, v_0) \in U$, and let $\delta > 0$ be such that the closed disc $R_\delta = \{(u,v) \in R^2 / (u-u_0)^2 + (v-v_0)^2 \le \delta^2\}$ withcentre(u_0, v_0) and radius δ is contained in U.Then prove that $\lim_{\delta \to 0} \frac{\mathcal{A}_N(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K|$,where K is the Gaussian curvature of σ at $\sigma(u_0, v_0)$.

b)Prove that a point **p** of a surface ***S***is an umbilic if and only if the Weingarten map $W_{\mathbf{p},S}$ is a scalar multiple of the identity map.

$(2 \times 5 = 10$ *weightage*$)$

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1M3N20201 (Pages : 2) Reg. No:.......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Degree Examination, November 2020

MMT3C12 – Complex Analysis

(2019 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A
Answer all questions
Each question carries 1 weightage

1. Find the fixed points of the linear transformation $w = \frac{2z}{3z-1}$.
2. Find the point at which the function $\tan z$ is not analytic.
3. State the symmetry principle.
4. If $z = x + iy$, prove that $|e^z| = e^x$.
5. Let n be a positive integer. Prove that $\int_\gamma (z-a)^n dz = 0$ for any closed curve γ.
6. Determine the nature of singularity of the function $\frac{\sin z}{z}$ at $z=0$. Justify your answer.
7. Find the residue of the function $f(z) = \frac{z^2-2}{(z-2)^2}$ at $z = 2$.
8. Define: Simply connected region. Give an example of a simply connected region.

(8 x 1 = 8 Weightage)

Part B
Answer any two questions from each unit
Each question carries 2 weightage

UNIT I

9. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$ then $g \circ f$ is analytic on G $(g \circ f)'(z) = g'(f(z))f'(z)$ for all z in G.
10. Prove that if γ is piecewise smooth and $f: [a,b] \to \mathbb{C}$ is continuous then $\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)dt$.
11. Prove that the cross ratio (z_1,z_2,z_3,z_4) is real if and only if the four points lie on a circle or a straight line.

UNIT II

12. Prove that if $p(z)$ is a non constant polynomial then there is a complex number a with $p(a) = 0$.

13. State and prove Morera's theorem.

14. Let G be a region and suppose that f is a non constant analytic function on G. then prove that for any open set U in G $f(U)$ is open.

UNIT III

15. Prove that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity.

16. State and prove Residue theorem.

17. Show that $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$

(6 x 2 = 12 weightage)

Part C

Answer any two questions
Each question carries 5 weightage

18. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius or convergence $R>0$ then prove that For each $k \geq 1, \sum_{n=k}^{\infty} n(n-1) \dots (n-k+1)a_n(z-a)^{n-k}$ the series has radius of convergence R and f is infinitely differentiable on $B(a;R)$.

19. State and prove Goursat's theorem

20. State and prove Cauchy's integral formula and evaluate $\int_{|z|=2} \frac{dz}{z^2+1}$

21. (a)Discuss the evaluation of integrals of the type $\int_{-\infty}^{\infty} R(x)\, e^{ix} dx$ using the residues.

    (b) Show that $\int_0^{\infty} \frac{sinx}{x}\ dx = \frac{\pi}{2}$

(2 x 5 = 10 weightage)

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Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Degree Examination, November 2020

MMT3C13 – Functional Analysis

(2019 Admission onwards)

Time: 3 hours Max. Weightage : 30

PART A

Answer all questions. Each question carries a weightage 1.

1. Prove or disprove : 'Every essentially bounded function is bounded'.
2. Prove that every norm on $\mathbb{K}$ is a positive scalar multiple of the absolute value norm.
3. Let X be normed space and f be a nonzero linear functional on X. If f be a discontinuous linear functional on X then prove that the zero space $Z(f)$ of f is dense in X.
4. Let X be a normed space over $\mathbb{K}$, $f \in X'$ *and* $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. If $U(a,r) \cap Z(f) = \emptyset$ then prove that $\|f\| \leq \frac{1}{r}$.
5. Prove that the subspace $C([a,b])$ of $L^p([a,b])$ consisting of all scalar valued continuous functions is not a Banach space.
6. What is the geometric interpretation of the Uniform boundedness principle.
7. Let X and Y be normed spaces and $F: X \to Y$ be a continuous map. Prove that F is a closed map.
8. Let x_1 *and* x_2 be two orthogonal elements in an inner product space X. Prove that $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$.

(8×1=8 Weightage)

PART B

Answer any two questions from each unit
Each question carries a weightage 2.

UNIT I

9. If $1 \le p < \infty$, prove that the set of all simple measurable functions on a measurable set Ewhich are zero outside subsets of finite measure is dense in $L^p\ (E)$.

10. Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space ${}^{X}/_{Y}$, let$|||x + Y||| = inf\{||x + y||: y \in Y\}$. Then prove that$|||\ .\ |||$ is norm on the quotient space ${}^{X}/_{Y}$

11. Show by an example that not all linear functionals on a normed space X are continuous.

(2×2 = 4 Weightage)

UNIT II

12. Let Y be a subspace of a normed space X and$a \in X\ but\ a \notin \bar{Y}$. Then prove that there is an $f \in X'$ such that$f(y) = 0\ for\ every\ y \in Y, f(a) = dist(a, \bar{Y})$ and$||f|| = 1$.

13. If every absolutely summable series is summable in a normed space X then prove that X is a Banach space.

14. Let X be a normed space and E be a subset of X.Prove that E is bounded in X if $f(E)$ is bounded in $\mathbb{K}$ for every $f \in X'$.

(2×2 = 4 Weightage)

UNIT III

15. Let X be a normed space and $P: X \to X$ be a projection. Suppose $R(P)$ and $Z(P)$are closed in X then prove that P is closed.

16. State and prove the parallelogram law for inner product spaces.

17. Let $\{u_1, u_2, \dots\}$be a countable orthonormal set in an inner product space $X$with the inner product $\langle\ ,\ \rangle$and $x \in X$. Then prove that $\sum_{i=1}^{n} |\langle x, u_i\rangle|^2 \le ||x||^2$.

(2×2 = 4 Weightage)

Part C

Answer any two question. Each question carries a weightage 5

18. (a) Prove that every closed and bonded subset of a finite dimensional normed space X is compact.

(b) Let X and Y be normed spaces and $F: X \to Y$ be linear then prove that F iscontinuous if and only if for every Cauchy sequence $\{x_n\}$in X the sequence$\{F(x_n)\}$ is Cauchy in Y.

(c) let X= $\mathbb{K}^2$ with the norm$\|.\|_\infty$. Let $Y = \{(x(1), x(2)) : x(2) = 0\}$and $g \in X'$ be such that $g(x(1), x(2)) = x(1)$.Find the Hahn Banach extension of g.

19. (a)Prove that every nonzero linear functional on a normed space X is open.

(b)Let X be a normed space, Y be a Banach space and $F_n \in BL(X, Y)$ be such that $\|F_n\| \leq \alpha$ for all n and for some$\alpha > 0$. Let E be a subset of X whose span is dense in X. Suppose that $(F_n(x))$ converges in Y for every x in E. Then prove that there is a unique $F \in BL(X, Y)$ such that $F_n(x) \to F(x)$ for every $x \in X$.

(c)State and prove the bounded inverse theorem.

20. (a)Let X be a normed space and Y be a closed proper subspace of X. Let r be a real number such that $0 < r < 1$. Then prove that there exists an $x_r \in$ X such that $\|x_r\| = 1$ and $r \leq dist(x_r, Y) \leq 1$

(b)Let X *and* Y be Banach spaces and $F: X \to Y$ be a closed linear map. Then prove that F is continuous.

21. (a)State and prove the Open Mapping Theorem..

(b)Let $\langle \, , \, \rangle$be an inner product on a linear space $X$and $T : X \to X$ be a linear one to one map. Let $\langle x, y \rangle_T = \langle Tx, Ty \rangle$, $x, y \in$ X . Then prove that $\langle \, , \, \rangle$ is an inner product on X.

(2×5 = 10 Weightage)

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Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Degree Examination, November 2020

MMT3C14 – PDE & Integral Equations

(2019 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A

Answer all questions. Each question carries 1 weightage

1. For the initial value problem $u_x + u_y = 1,\ u(x,0) = f(x)$, what are the projections of the characteristic curves on the (x, y) plane?
2. Consider the equation $xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$. Find the domain where the equation is elliptic, and the domain where it is hyperbolic.
3. If $u(x,t)$ is the solution of the Cauchy problem

$$u_{tt} - u_{xx} = 0\,;\, 0 < x < \infty, t > 0,$$
$$u(0,t) = t^2\,;\, t > 0,$$
$$u(x,0) = x^2\,; 0 \le x < \infty,$$
$$u_t(x,0) = 6x\,;\, 0 \le x < \infty,$$

evaluate $u(4,1)$.

4. Explain Separated and Periodic boundary conditions in heat conduction problems.
5. Let $u(x,y)$ be a harmonic function in a domain D, show that $u \in C^\infty(D)$.
6. Reduce the Volterra integral equation $y(x) = x - cosx + \int_0^x (x - \xi)y(\xi)\, d\xi$ to equivalent initial value problem.
7. Define separable kernel. Is $e^{x\xi}$ separable? Justify your answer.
8. Determine the resolvent kernel associated with the kernel $K(x,\ \xi) = x\xi$ in the interval $(0,1)$ in the form of a power series in λ.

(8 x 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each question carries 2 weightage

Unit I

9. Find a compatibility condition for the Cauchy problem

$$u_x{}^2 + u_y{}^2 = 1, \qquad u(\cos s, \sin s) = 0, \qquad 0 \le s \le 2\pi$$

Also solve the problem.

10. If the equation $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$, where $a, b, \dots, f, g$ are given functions of x and y, is hyperbolic in a domain D, then show that there exists a coordinate system (ξ, η) in which the equation has the canonical form $w_{\xi\eta} + l_1[w] = G(\xi, \eta)$, where $w(\xi, \eta) = u\big(x(\xi, \eta), y(\xi, \eta)\big)$, l_1 is a first-order linear differential operator, and G is a function which depends on the given PDE.

11. Show that the Cauchy problem

$$u_{tt} - c^2 u_{xx} = F(x, t)\ ;\ -\infty < x < \infty, t > 0$$
$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x)\ ;\ -\infty < x < \infty,$$

admits at most one solution.

Unit II

12. Solve the problem

$$u_t - u_{xx} = 0\ ; \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(0, t) = u(\pi, t) = 0\ ;\ \ t \ge 0$$
$$u(x, 0) = \begin{cases} x & ;\ 0 \le x \le \pi/2 \\ \pi - x & ;\ \pi/2 \le x \le \pi \end{cases}$$

13. Solve the Laplace equation $\Delta u = 0$ in the square $0 < x, y < \pi$, subject to the boundary condition $u(x, 0) = u(x, \pi) = 1,\ u(0, y) = u(\pi, y) = 0$.

14. Outline the energy method. Demonstrate the energy method for the Neumann problem for the vibrating string.

Unit III

15. For the homogeneous Fredholm integral equation $y(x) = \lambda \int_a^b K(x,\xi)\, y(\xi)\, d\xi$, with symmetric kernel $K(x,\xi)$, show that the characteristic functions corresponding to distinct characteristic numbers are orthogonal over (a,b).

16. Determine the characteristic values and characteristic functions for the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} cos(x+\xi)\, y(\xi)\, d\xi$$

17. Write a note on Neumann series.

(6 x 2 = 12 weightage)

Part C

Answer any two questions. Each question carries 5 weightage

18. (a) State and prove the existence and uniqueness theorem for the Cauchy problem of first order Quasilinear equations.

(b) Convert the equation $u_{xx} + 4u_{xy} + u_x = 0$ into a canonical form and hence find its general solution.

19. (a) Use the method of separation of variables to solve linear homogeneous initial boundary value problem for the wave equation.

(b) State and prove The weak maximum principle and The strong maximum principle.

20. (a) Discuss the method of solving boundary value problems using the Green's function, for the equation $p(x)\, y'' + p'(x) y' + q(x)\, y + \Phi(x) = 0$ with homogeneous boundary conditions $\alpha y + \beta y' = 0$ at the end points of the interval $a \le x \le b$.

(b) Transform the boundary value problem $y'' + xy = 1\,; y(0) = y(1) = 0$ to the corresponding Integral equation.

21. (a) Analyze the problem $xuu_x + yuu_y = x^2 + y^2\,;\ x > 0,\ y > 0,\ u(x,1) = \sqrt{x^2+1}$. using Lagrange method. Determine whether there exists a unique solution, infinitely many solutions or no solution at all. If there is a unique solution, find it; if there are infinitely many solutions, find at least two of them.

(b) Derive the formula $\underbrace{\int_a^x \ldots \int_a^x}_{n\ times} f(x)\, dx \ldots dx = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi$

(2 x 5 = 10 weightage)

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Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Degree Examination, November 2020

MMT3E03 – Measure & Integration

(2019 Admission onwards)

Time: 3 hours

Max. Weightage : 30

Section A

Answer ALL questions. Each question carries 1 weight.

1. Does there exist an infinite σ-algebra which has only countably many members ?

2. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be sets in $\mathfrak{M}$.

   If $A = A_1 \cup A_2 \cup A_3 \cup \cdots$, then prove that $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$.

3. Define Lebesgue integral of a measurable function over a set in a σ-algebra.

   If $A \subseteq B$ and $f \geq 0$, then prove that $\int_A f\, d\mu \leq \int_B g\, d\mu$.

4. Let μ be the counting measure on the set of integers and let $E = \{1, 2, \cdots, N\}$.

   If $f(x) = x$ is defined on E, find $\int_E f\, d\mu$.

5. Define a complex measure μ and its total variation measure $|\mu|$. Prove that $|\mu(E)| \leq |\mu|(E)$.

6. Define absolute continuity of measures.

   Let λ_1 and λ_2 be measures and μ be a positive measure.

   If $\lambda_1 << \mu$ and $\lambda_2 << \mu$, then prove that $(\lambda_1 + \lambda_2) << \mu$.

7. Explain the Radon-Nykodym derivative with all necessary details.

8. Define "Monotone Class" and give one example. **$8\times1 = 8$ Weights.**

Section B

Answer any TWO questions from each unit. Each question carries 2 weights.

UNIT I

9. Define a measurable function. Let X be the set of all integers and $\mathfrak{M} = \{X, E, F, \Phi\}$ where E is the set of positive integers and F is the set of all integers ≤ 0. Let $f : X \longrightarrow \mathbf{R}$ be defined as $f(x) = \cos \pi x$. Verify whether f is measurable or not.

10. State and prove Lebesgue's Monotone Convergence Theorem.

11. Explain the concept of "a property almost everywhere" with respect to a measure.

Let $(X, \mathfrak{M}, \mu)$ be a measure space. If $f = g$ a.e. on X, prove that $\int f\,d\mu = \int g\,d\mu$.

UNIT II

12. Let X be a locally compact, σ-compact Hausdorff space. Let $\mathfrak{M}$ be a σ-algebra containing all Borel subsets of X. Let μ be a regular Borel measure on $\mathfrak{M}$. If $E \in \mathfrak{M}$, prove that there is an F-σ set A and a G-δ set B such that $A \subset E \subset B$ and $\mu(B - A) = 0$.

13. For eah $n = 1, 2, 3, \cdots$ consider P_n as the set of all $x \in R^k$ whose co-ordinates are integral multiples of 2^{-n} and Ω_n as the collection of all 2^{-n} boxes with corners at points of P_n. Prove that every non-empty open set in R^k is a countable union of disjoint boxes belonging to $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \cdots$.

14. Prove that the total variation measure of a complex measure is a positive measure.

UNIT III

15. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ be measure spaces and f be an $\mathscr{S} \times \mathscr{T}$-measurable function on $X \times Y$. Prove that for each $x \in X$ the function f_x defined as $f_x(y) = f(x,y)$ is a $\mathscr{T}$-measurable on Y.

16. State Fubini's theorem.

Do the existence of both the iterated integrals guarantee the conclusion of Fubini's theorem ?

Give reason.

17. Let m_k denote the Lebesgue measure on R^k. If $k = r + s,\ r \geq 1,\ s \geq 1$. Then prove that m_k is the completion of the product measure $m_r \times m_s$.

$6 \times 2 = 12$ Weights.

Section C

Answer any TWO questions. Each question carries 5 weights.

18. a) Let f be a measurable simple function on a set X, $\mathfrak{M}$ a σ-algebra on X, $E \in \mathfrak{M}$ and μ a measure on $\mathfrak{M}$. Prove that $\phi(E) = \int_E f\,d\mu$ is a measure on $\mathfrak{M}$.

b) State and prove Fatou's Lemma.

c) Give one example to show that strict inequality can hold in Fatou's Lemma.

. State and prove Vitali-Carathéodory theorem.

. State and prove the Hahn decomposition theorem.

. Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be measurable spaces.

a) Define the terms **measurable rectangle**, **elementary sets**, **monotone class**, x**-section** and y**-section**.

b) If $E \in \mathscr{S} \times \mathscr{T}$, then prove that $E_x \in \mathscr{T}$ and $E^y \in \mathscr{S}$.

c) Prove that $\mathscr{S} \times \mathscr{T}$ is the smallest monotone class which contains all elementary sets.

$2 \times 5 = 10$ Weights.