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1M3N19202 (Pages : 3) Reg. No:..........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc. Mathematics Degree Examination, November 2019

MT3C14 – Functional Analysis

(2018 Admission onwards)

Time: 3 hours Max. Weightage: 36

PART A

Answer *all* questions (Each question has weightage 1.)

1. Show that $d_\infty(x,y) = \sup_{t\in T}|x(t)-y(t)|$ defines a metric on the set of all *K*-valued bounded functions on some set *T*.
2. Is $C([0,1])$ with *p*-norm a Banach space? Justify.
3. Show that ℓ^∞ is not separable.
4. Give an example to show that a subspace of an infinite dimensional normed space need not be closed.
5. If *X* is a normed space and (x_n) is a Cauchy sequence in *X*, then show that the scalar sequence $(\|x_n\|)$ converges.
6. Let *X* be a normed space and $x, y \in X$, $r > 0$. Then prove that $U(x + y, r) = U(x,r) + y$.
7. If *X* is a normed space over $K, f \in X'$ and $f \neq 0$ and if $a \in X$ with $f(a) = 1$ and $r > 0$. Then show that $U(a, r) \cap Z(f) = \phi$ if and only if $\|f\| \leq \frac{1}{r}$.
8. If all bounded linear functionals on a normed space *X* vanishes at a given point *x* of *X*, then prove that *x* must be zero.
9. How a normed space *X* can be viewed as a subspace of its second dual X''?
10. Define Schauder basis. Give one example.
11. Give an example to show that the completeness condition on *X* cannot be dropped in the uniform boundedness principle

12. Give an example for a closed, linear map which is not continuous.

13. Show that among all the p- norms , $\|.\|_p$ $1 \le p \le \infty$, on K^n $(n \ge 2)$, only the norm $\|.\|_2$ is induced by an inner product.

14. Show that the set $\left\{ \frac{e^{\text{int}}}{\sqrt{2\pi}} ; n = 0, \pm 1, \pm 2, ... \right\}$ is an orthonormal set in L^2 ([-π, π]) .

(14 × 1 = 14 weightage)

PART B

Answer any *seven* questions (Each question has weightage 2.)

15. Show that the metric space ℓ^∞ is complete.

16. If $\|.\|$, $\|.\|'$ are any two norms on a linear space X, then prove that the norm, $\|.\|$ is stronger than $\|.\|'$ if and only if there exists $\alpha > 0$ such that $\|x\|' \le \alpha \|x\| \quad \forall x \in X$.

17. For $F \in BL(X, Y)$, define the operator norm, and show that

$$\|F\| = \inf \left\{ \alpha \ge 0;\ \|F(x)\| \le \alpha \|x\| \quad for \quad all \quad x \in X \right\}.$$

18. Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Show that F is continuous on X if and only if $\|F(x)\| \le \alpha \ \|x\|$ for all $x \in X$ and for some $\alpha > 0$.

19. Let X and Y be normed spaces and $X \ne \{0\}$. Then show that BL(X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

20. Prove that a Banach space cannot have a denumerable Hamel basis.

21. Let X and Y be normed spaces and $F: X \to Y$ be linear. Then prove that F is continuous if and only if $g \circ F$ is continuous, $\forall\, g \in Y'$.

22. If X is a normed space and if $P : X \to X$ is a projection, then show that P is a closed map if and only if the subspaces R(P) and Z(P) are closed in X.

23. Let X and Y be two inner product spaces and $F : X \to Y$ be linear. Show that

$$\|F(x)\| = \|x\|, \forall x \in X \Leftrightarrow \langle F(x_1), F(x_2) \rangle = \langle x_1, x_2 \rangle, \forall x_1, x_2 \in X$$

24. If $\{u_\alpha\}$ is an orthonormal set in an inner product space X, then for any $x \in X$, show that the set $E_x = \{u_\alpha ; \langle x, u_\alpha \rangle \ne 0\}$ is a countable set.

(7 × 2 = 14 weightage

PART C

Answer any *two* questions (Each question has weightage 4.)

25. If E is a measurable subset of R, prove that the spaces (L^p(E), $\|.\|_p$), $1 \leq p \leq \infty$ are all Banach spaces.

26. State Hahn – Banach extension theorm. Show that for every subspace Y of a normed space X and every g in Y' there is a unique Hahn-Banach extension of g to X if and only if X' is strictly convex.

27. If X and Y are Banach spaces and if $F : X \rightarrow Y$ is a closed linear map, then prove that F is continuous.

28. (a) State and prove Schwarz inequality.
    (b) State and prove Bessel's inequality.

($2 \times 4 = 8$ weightage)

1M3N19203 (Pages : 3) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc. Mathematics Degree Examination, November 2019

MT3C15 – PDE and Integral Equations

(2018 Admission onwards)

Time: 3 hours Max. Weightage : 36

Section A

Answer ALL questions. Each question carries 1 weight.

1. If $P(t) : (x(t), y(t), z(t))$ is a point on the surface $\mathbf{S} : z = F(x,y)$, then show that $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -1\right)$ will be the direction ratios of the normal to the surface at the point $P(t)$.

2. If F is an arbitrary function, obtain the *pde* corresponding to $z = x + y + F(x^2 + y^2)$.

3. Define the different types of first order partial differential equations.

   Give example of a *semilinear pde* that is not *linear*.

4. Show that $(x-a)^2 + (y-b)^2 + z^2 = 1$ is a complete integral of $z^2(1 + p^2 + q^2) = 1$.

5. Show that the Pfaffian differential equation $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ is integrable.

6. Write the complete integral of the *pde* $z = px + qy + p^3 + 2q^5$.

   Check that it is a complete integral.

7. Describe the method to find the integral surface through a given curve corresponding to a quasilinear partial differential equation.

8. What is meant by Monge Cone at a point (x_0, y_0, z_0) corresponding to a non-linear first order partial differential equation $f(x, y, z, p, q) = 0$?

9. Identify the type of the equation $u_{xx} + 2u_{xy} + 17u_{yy} = 0$.

10. Describe the problem of vibration of a string of finite length along with the boundary conditions.

11. State the Maximum and Minimum principles correponding to a harmonic function.

Explain all terms used in the statement.

12. Prove that $\int_a^x \int_a^{x_n} \cdots \int_a^{x_3} int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{(n-1)} dx_n = \frac{1}{(n-1)!}\int_a^x (x-\zeta)^{(n-1)} f(\zeta)d\zeta$.

13. Describe the different types of integral equations. Give one example for each type.

14. State and prove the Abel's formula.

Section B

Answer any SEVEN questions. Each question carries 2 weights.

15. Prove that singular solution of a *pde* $f(x,y,z,p,q)=0$ is obtained by eliminating p and q from the equations $f(x,y,z,p,q)=0$, $f_p(x,y,z,p,q)=0$ and $f_q(x,y,z,p,q)=0$.

16. Find the general integral of $(z^2-2yz-y^2)p+x(y+z)q=x(y-z)$.

17. Show that the equations $xp-yq=x$ and $x^2p+q=xz$ are compatible and find a one parameter family of common solutions.

18. Solve by Jacobi's method the equation $z+2u_z=(u_x+u_y)^2$.

19. For the differential equation $x(z+2)p+(xz+2yz+2y)q=z(z+1)$, find the integral surface passing through the curve $x_0=s^2$, $y_0=0$ and $z_0=2s$.

20. Obtain the d'Alembert's solution of the problem of vibrations of an infinite string.

21. Describe the Neumann problem.

Prove that solution of the Neumann problem is unique upto the addition of a constant.

22. If $y''(x)=F(x)$ and y satisfies the initial conditions $y(0)=y_0$ and $y'(0)=y'_0$, then show that $y(x)=\int_0^x (x-\zeta)F(\zeta)\,d\zeta+y'_0x+y_0$.

23. What is meant by characteristic values of a homogeneous Fredholm Integral Equation ?

Prove that the characteristic functions corresponding to two different characteristic values are orthogonal.

. Find the resolvant kernel where the kernel of the integral equation is $K(x,\zeta) = 1 - 3x\zeta$ in the interval $(0,1)$.

Section C

Answer any TWO questions. Each question carries 4 weights.

5. What is meant by the compatibility of two non-linear first order partial differential equations $f(x,y,z,p,q) = 0$ and $g(x,y,z,p,q) = 0$.

   Describe the Charpit's method to find the complete integral of a first order partial differential equation. Find the complete integral of $p+q-pq=0$.

6. Explain the concept of initial strip corresponding to a partial differential equation and an inital data curve. Find the solution of the equation $z = \frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ which passes through the x-axis.

7. Discuss the problem of heat conduction in a finite rod.

   Solve $u_t = u_{xx}, \quad 0 < x < l, \quad t > 0$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and the initial condition $u(x,0) = x(l-x), \quad 0 \leq x \leq l$.

8. For the integral equation $y(x) = \lambda \int_0^1 (1-3x\zeta)y(\zeta)d\zeta + F(x)$, write the *kernel*.

   Solve the equation by the method of separable kernel.

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N19200 (Pages : 2) Reg. No:...........................

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FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc. Mathematics Degree Examination, November 2019

MT3C12 – Multivariable Calculus & Geometry

(2018 Admission onwards)

e: 3 hours Max. Weightage: 36

Part A

***Answer all questions* $(1-14)$**

***Each question has weightage* 1**

If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, show that A is uniformly continuous.

Let $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{f}(x,y) = (e^x \cos y, e^x \sin y)$. Find $\left[\mathbf{f}'\left(0, \frac{\pi}{2}\right)\right]$.

If $A \in L(X,Y)$ and $B \in L(Y,Z)$, where X, Y and Z are vector spaces, prove that

$[BA] = [B][A]$.

State the implicit function theorem.

Does the curve $\gamma(t) = (\cos^2 t, \sin^2 t)$, $t \in \mathbb{R}$, has unit speed reparametrization? Explain.

If γ is a unit-speed curve, then prove that $\ddot{\gamma}$ is perpendicular to $\dot{\gamma}$.

Show that any reparametrization of a regular curve is regular.

Give a parameterization of the parabola $y = x^2$ which is not regular.

Find the curvature of the curve $\gamma(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right)$.

Prove that any open disc in the xy −plane is a surface.

If γ is a curve lying in the image of a surface patch $\boldsymbol{\sigma}$, prove that $\langle \dot{\gamma}, \dot{\gamma} \rangle = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$.

Prove that the second fundamental form of a plane in $\mathbb{R}^3$ is zero.

If γ is a unit-speed curve on an oriented surface, prove that $\kappa_n = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle$.

Give an example of a surface of which every point is an umbilic.

$(\mathbf{14 \times 1 = 14}\ \boldsymbol{weightage})$

Part B

Answer any seven questions $(15-24)$
Each question has weightage 2

15. Prove that a linear operator A on a finite dimensional vector space X is one-to-one if and o the range of A is X.
16. Let $\mathbf{\Omega}$ be the set of all linear operators in $\mathbb{R}^n$. Prove that $\mathbf{\Omega}$ is an open subset of $L(\mathbb{R}^n)$.
17. Suppose $\mathbf{f}$ maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, $\mathbf{f}$ is differentiable in E and there is a number M such that $\|\mathbf{f}'(\mathbf{x})\| \le M$, for every $\mathbf{x} \in E$. Prove that $|\mathbf{f}(b) - \mathbf{f}(a)| \le |b-a|$ f $\boldsymbol{a}, \boldsymbol{b} \in E$.
18. Find the unit speed reparametrization of the curve $\gamma(t) = (e^t \cos t, e^t \sin t)$.
19. Let γ be a regular closed curve. Prove that a unit speed reparametrization of γ is closed.
20. Compute the torsion of the curve $\gamma(t) = (a \cos t, a \sin t, bt)$; $t \in \mathbb{R}$, a and b are cons
21. Prove that the unit cylinder $\boldsymbol{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a surface.
22. Compute the first fundamental form of the surface $\boldsymbol{\sigma}(\theta, \varphi) = (\cos\theta\, cos\varphi, \cos\theta \sin\varphi$, si
23. State and prove Euler's theorem.
24. Define the elliptic, hyperbolic and parabolic points of a surface.

$(7 \times 2 = 14$ ***weightage***$)$

Part C

Answer any two questions $(25-28)$
Each question has weightage 4

25. Suppose $\mathbf{f}$ maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that $\mathbf{f} \in \mathcal{C}'(E)$ if and only if the part derivatives $D_j \mathrm{f}_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.
26. State and prove contraction principle.
27. a)Prove that the total signed curvature of a closed plane curve is an integer multiple of 2

    b)Let γ be a curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Prove that γ is a parametrization of (part of) a circle.
28. a) Let $\boldsymbol{\sigma}$ be a surface patch of an oriented surface $\boldsymbol{S}$. With the usual notations, prove tha matrix of Weingarten map with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ is $\mathcal{F}_1^{-1}\mathcal{F}_{11}$.

    b) Find the principal curvatures and the corresponding principal vectors of the cylinder $\sigma(u, v) = (\cos v, \sin v, u)$.

$(2 \times 4 = 8$ ***weig***

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1M3N19201 (Pages : 2) Reg. No:............................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc. Mathematics Degree Examination, November 2019

MT3C13 – Complex Analysis

(2018 Admission onwards)

Time: 3 hours Max. Weightage: 36

Part A

Answer all questions

Each question carries 1 weightage

1. Map the left half plane $Re\ z < 0$ onto the unit disc $|z| < 1$.
2. If $T_1z = \frac{z+2}{z+3}$ and $T_2z = \frac{z}{z+1}$ find $T_1^{-1}T_2z$.
3. What is the necessary condition for a mapping to be conformal.
4. Find the cross ratio of $(1, -1, i, -i)$.
5. If $z = x + iy$, prove that $|e^z| = e^x$.
6. Prove that an integral over an arc depends only on the end points if the integral on an any closed curve is zero.
7. Prove that $(C, a) = 0$, C is a circle and a is any point outside C.
8. Identify the singularity of $\frac{z}{cosz}$
9. Find the residue of the function $f(z) = \frac{z^2-2}{(z-2)^2}$ at $z = 2$.
10. If f is analytic prove that $\ln(|f|)$ is harmonic.
11. Find the Taylor series expansion of the function $\frac{1}{z}$ *about* $z = i$
12. Define a unimodular transformation
13. Prove that an elliptic function without poles is constant.
14. Prove that Weierstrass elliptic function $\mathcal{P}(z)$ is even. (14 X 1 = 14 Wt)

Part B

Answer any seven questions

Each question carries 2 weightage

15. Prove that the set of all linear transformation form a group under the operation composition.

16. Prove that cross ratio is invariant under bilinear transformation.

17. If z and z^* are symmetric points with respect to the circle C where $C = \{z : |z-a| = R, \; 0 < R < \infty\}$ show that $(z^* - a)(\bar{z} - \bar{a}) = R^2$.

18. Evaluate $\int_{|z|=2} \frac{1}{z^2+1} \, dz$.

19. State and prove Fundamental Theorem of Algebra.

20. Prove that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity.

21. Prove that a non constant harmonic function has neither maximum nor minimum on a region of definition. Consequently the maximum and minimum on a closed bounded set E are taken on the boundary of E.

22. State and prove Rouche's theorem.

23. Evaluate $\int_0^\infty \frac{1}{1+x^2} dx$

24. Prove that a non constant elliptic function has equally many poles as it has zeros

(7 X 2 = 14 Wt)

Part C

Answer any two questions

Each question carries 4 weightage

25. Suppose that $\emptyset(\zeta)$ is continuous on an arc γ. Then the function $F_n(z) = \int_\gamma \frac{\emptyset(\zeta)\, d\zeta}{(\zeta - z)^n}$ is analytic in each of the region determined by γ and its derivative is $F_n^{'}(z) = n\, F_{n+1}(z)$

26. Discuss the evaluation of integrals of the type $\int_{-\infty}^{\infty} R(x)\, e^{ix} dx$ using the residues.

27. State and prove Cauchy's General Theorem.

28. Show that the function $\mathcal{P}(z)$ satisfies an equation of the form

$$\mathcal{P}'(z)^{\;2} = 4\,\mathcal{P}(z)^{\;3} - g_2\mathcal{P}(z) - g_3.$$

(2 X 4 = 8 Wt)