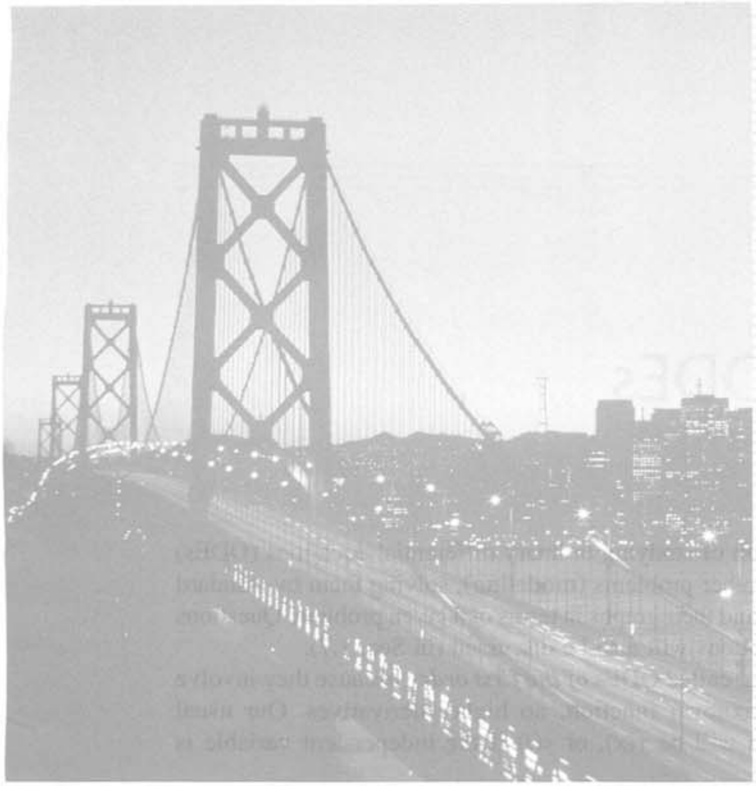


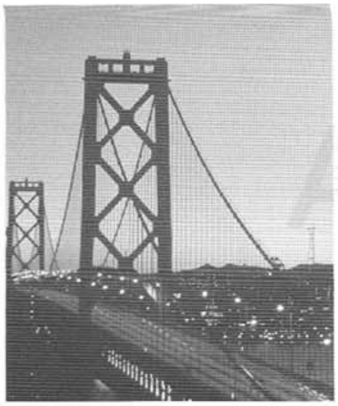
PART A

Ordinary Differential Equations (ODEs)

Differential equations are of basic importance in engineering mathematics because many physical laws and relations appear mathematically in the form of a differential equation. In Part A we shall consider various physical and geometric problems that lead to differential equations, with emphasis on ***modeling,*** that is, the transition from the physical situation to a "mathematical model." In this chapter the model will be a differential equation, and as we proceed we shall explain the most important standard methods for solving such equations.

Part A concerns ***ordinary*** **differential equations (ODEs),** whose unknown functions depend on a *single* variable. *Partial* differential equations (PDEs), involving unknown functions of *several* variables, follow in Part C.

ODEs are very well suited for computers. ***Numeric methods for ODEs can be studied directly after Chaps.* 1 *or* 2.** See Secs. 21.1–21.3, which are independent of the other sections on numerics.



CHAPTER 1

First-Order ODEs

In this chapter we begin our program of studying ordinary differential equations (ODEs) by deriving them from physical or other problems **(modeling),** solving them by standard methods, and interpreting solutions and their graphs in terms of a given problem. Questions of existence and uniqueness of solutions will also be discussed (in Sec. 1.7).

We begin with the simplest ODEs, called ODEs ***of the first order*** because they involve only the first derivative of the unknown function, no higher derivatives. Our usual notation for the unknown function will be $y(x)$, or $y(t)$ if the independent variable is time t.

If you wish, use your computer algebra system (CAS) for checking solutions, but make sure that you gain a conceptual understanding of the basic terms, such as ODE, direction field, and initial value problem.

COMMENT. ***Numerics for first-order ODEs can be studied immediately after this chapter.*** See Secs. 21.1–21.2, which are independent of other sections on numerics.

Prerequisite: Integral calculus.
Sections that may be omitted in a shorter course: 1.6, 1.7.
References and Answers to Problems: App. 1 Part A, and App. 2

1.1 Basic Concepts. Modeling

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, equations, and so forth. Such an expression is known as a mathematical **model** of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling.** We shall illustrate this process by various examples and problems because modeling requires experience. (Your computer may help you in solving but hardly in setting up models.)

Since many physical concepts, such as velocity and acceleration, are derivatives, a model is very often an equation containing derivatives of an unknown function. Such a model is called a **differential equation.** Of course, we then want to find a solution (a function that satisfies the equation), explore its properties, graph it, find values of it, and interpret it in physical terms so that we can understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution we must first define basic concepts needed throughout this chapter.

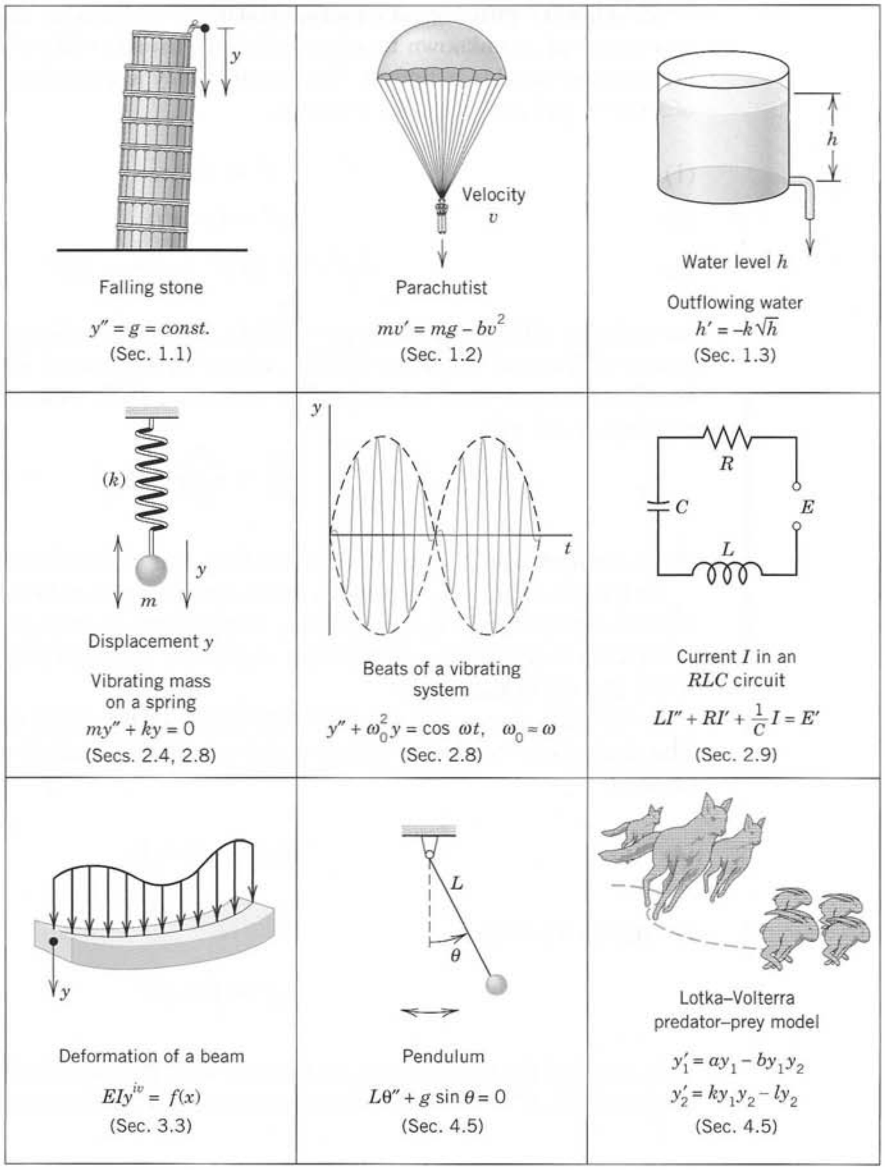


Fig. 1. Some applications of differential equations

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

$$y' = \cos x, \tag{1}$$

$$y'' + 9y = 0, \tag{2}$$

$$x^2 y''' y' + 2e^x y'' = (x^2 + 2)y^2 \tag{3}$$

are ordinary differential equations (ODEs). The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDEs are more complicated than ODEs; they will be considered in Chap. 12.

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs.** Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

$$F(x, y, y') = 0 \tag{4}$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form,* in contrast with the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

EXAMPLE 1 Verification of Solution

$y = h(x) = c/x$ (c an arbitrary constant, $x \neq 0$) is a solution of $xy' = -y$. To verify this, differentiate, $y' = h'(x) = -c/x^2$, and multiply by x to get $xy' = -c/x = -y$. Thus, $xy' = -y$, the given ODE. ■

EXAMPLE 2 Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions.*** Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 2 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$. ■

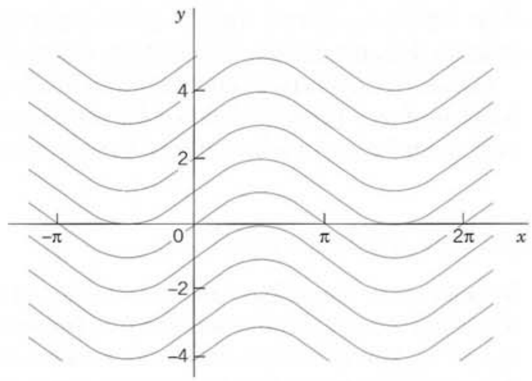


Fig. 2. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

EXAMPLE 3 Exponential Growth, Exponential Decay

From calculus we know that $y = ce^{3t}$ (c any constant) has the derivative (chain rule!)

$$y' = \frac{dy}{dt} = 3ce^{3t} = 3y.$$

This shows that y is a solution of $y' = 3y$. Hence this ODE can model **exponential growth,** for instance, of animal populations or colonies of bacteria. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as ***Malthus's law.***[1] We shall say more about this topic in Sec. 1.5.

Similarly, $y' = -0.2y$ (with a minus on the right!) has the solution $y = ce^{-0.2t}$. Hence this ODE models **exponential decay,** for instance, of a radioactive substance (see Example 5). Figure 3 shows solutions for some positive c. Can you find what the solutions look like for negative c? ■

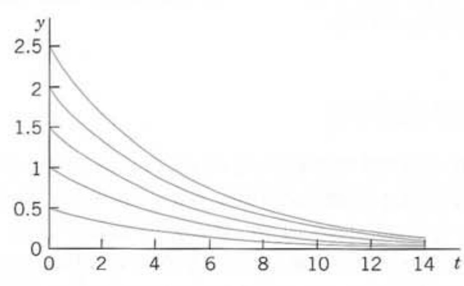


Fig. 3. Solutions of $y' = -0.2y$ in Example 3

[1]Named after the English pioneer in classic economics, THOMAS ROBERT MALTHUS (1766–1834).

We see that each ODE in these examples has a solution that contains an arbitrary constant c. Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

(We shall see that c is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.)

We shall develop methods that will give general solutions *uniquely* (perhaps except for notation). Hence we shall say ***the*** general solution of a given ODE (instead of ***a*** general solution).

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c. If we choose a specific c (e.g., $c = 6.45$ or 0 or -2.01) we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

In most cases, general solutions exist, and every solution not containing an arbitrary constant is obtained as a particular solution by assigning a suitable value to c. Exceptions to these rules occur but are of minor interest in applications; see Prob. 16 in Problem Set 1.1.

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE together with an initial condition is called an **initial value problem.** Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{5}$$

EXAMPLE 4 Initial Value Problem

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7.$$

Solution. The general solution is $y(x) = ce^{3x}$; see Example 3. From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

Modeling

The general importance of modeling to the engineer and physicist was emphasized at the beginning of this section. We shall now consider a basic physical problem that will show the typical steps of modeling in detail: Step 1 the transition from the physical situation (the physical system) to its mathematical formulation (its mathematical model); Step 2 the solution by a mathematical method; and Step 3 the physical interpretation of the result. This may be the easiest way to obtain a first idea of the nature and purpose of differential equations and their applications. Realize at the outset that your ***computer*** (your ***CAS***) may perhaps give you a hand in Step 2, but Steps 1 and 3 are basically your work. And Step 2 requires a solid knowledge and good understanding of solution methods available to you—*you* have to choose the method for your work by hand or by the computer. Keep this in mind, and always check computer results for errors (which may result, for instance, from false inputs).

EXAMPLE 5 Radioactivity. Exponential Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Physical Information. Experiments show that at each instant a radioactive substance decomposes at a rate proportional to the the amount present.

Step 1. Setting up a mathematical model (a differential equation) of the physical process. Denote by $y(t)$ the amount of substance still present at any time t. By the physical law, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. Denote the constant of proportionality by k. Then

$$\frac{dy}{dt} = ky. \tag{6}$$

The value of k is known from experiments for various radioactive substances (e.g., $k = -1.4 \cdot 10^{-11}\text{sec}^{-1}$, approximately, for radium ${}_{88}\text{Ra}^{226}$). *k is negative* because $y(t)$ decreases with time. The given initial amount is 0.5 g. Denote the corresponding time by $t = 0$. Then the initial condition is $y(0) = 0.5$. This is the instant at which the process begins; this motivates the term *initial condition* (which, however, is also used more generally when the independent variable is not time or when you choose a t other than $t = 0$). Hence the model of the process is the **initial value problem**

$$\frac{dy}{dt} = ky, \qquad y(0) = 0.5. \tag{7}$$

Step 2. Mathematical solution. As in Example 3 we conclude that the ODE (6) models exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(t) = ce^{kt}. \tag{8}$$

We now use the initial condition to determine c. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing this process is

$$y(t) = 0.5e^{kt} \qquad \text{(Fig. 4)}. \tag{9}$$

Always check your result—it may involve human or computer errors! Verify by differentiation (chain rule!) that your solution (9) satisfies (7) as well as $y(0) = 0.5$:

$$\frac{dy}{dt} = 0.5ke^{kt} = k \cdot 0.5e^{kt} = ky, \qquad y(0) = 0.5e^{0} = 0.5.$$

Step 3. Interpretation of result. Formula (9) gives the amount of radioactive substance at time t. It starts from the correct given initial amount and decreases with time because k (the constant of proportionality, depending on the kind of substance) is negative. The limit of y as $t \to \infty$ is zero. ■

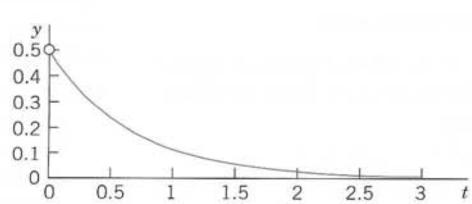


Fig. 4. Radioactivity (Exponential decay, $y = 0.5\, e^{kt}$, with $k = -1.5$ as an example)

EXAMPLE 6 A Geometric Application

Geometric problems may also lead to initial value problems. For instance, find the curve through the point (1, 1) in the xy-plane having at each of its points the slope $-y/x$.

Solution. The slope y' should equal $-y/x$. This gives the ODE $y' = -y/x$. Its general solution is $y = c/x$ (see Example 1). This is a family of hyperbolas with the coordinate axes as asymptotes.

Now, for the curve to pass through (1, 1), we must have $y = 1$ when $x = 1$. Hence the initial condition is $y(1) = 1$. From this condition and $y = c/x$ we get $y(1) = c/1 = 1$; that is, $c = 1$. This gives the particular solution $y = 1/x$ (drawn somewhat thicker in Fig. 5). ■

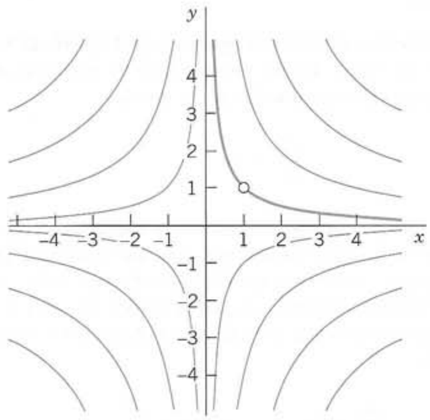


Fig. 5. Solutions of $y' = -y/x$ (hyperbolas)

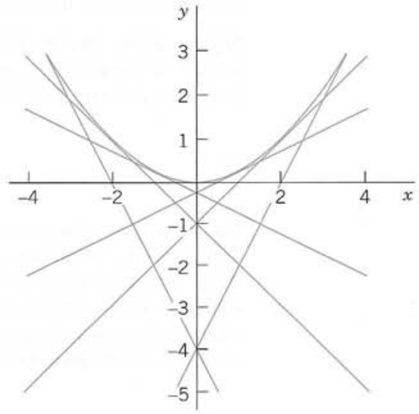


Fig. 6. Particular solutions and singular solution in Problem 16

PROBLEM SET 1.1

1–4 CALCULUS

Solve the ODE by integration.

1. $y' = -\sin \pi x$
2. $y' = e^{-3x}$
3. $y' = xe^{x^2/2}$
4. $y' = \cosh 4x$

5–9 VERIFICATION OF SOLUTION

State the order of the ODE. Verify that the given function is a solution. (a, b, c are arbitrary constants.)

5. $y' = 1 + y^2, \quad y = \tan(x + c)$
6. $y'' + \pi^2 y = 0, \quad y = a \cos \pi x + b \sin \pi x$
7. $y'' + 2y' + 10y = 0, \quad y = 4e^{-x} \sin 3x$
8. $y' + 2y = 4(x + 1)^2, \quad y = 5e^{-2x} + 2x^2 + 2x + 1$
9. $y''' = \cos x, \quad y = -\sin x + ax^2 + bx + c$

10–14 INITIAL VALUE PROBLEMS

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition. Sketch or graph this solution.

10. $y' = 0.5y, \quad y = ce^{0.5x}, \quad y(2) = 2$
11. $y' = 1 + 4y^2, \quad y = \frac{1}{2}\tan(2x + c), \quad y(0) = 0$
12. $y' = y - x, \quad y = ce^x + x + 1, \quad y(0) = 3$
13. $y' + 2xy = 0, \quad y = ce^{-x^2}, \quad y(1) = 1/e$
14. $y' = y \tan x, \quad y = c \sec x, \quad y(0) = \frac{1}{2}\pi$

15. **(Existence)** **(A)** Does the ODE $y'^2 = -1$ have a (real) solution?

    (B) Does the ODE $|y'| + |y| = 0$ have a general solution?

16. **(Singular solution)** An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of the kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

17–22 MODELING, APPLICATIONS

The following problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. **(Falling body)** If we drop a stone, we can assume air resistance ("drag") to be negligible. Experiments show that under that assumption the acceleration $y'' = d^2y/dt^2$ of this motion is constant (equal to the so-called acceleration of gravity $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$). State this as an ODE for $y(t)$, the distance fallen as a function of time t. Solve the ODE to get the familiar law of free fall, $y = gt^2/2$.

18. (Falling body) If in Prob. 17 the stone starts at $t = 0$ from initial position y_0 with initial velocity $v = v_0$, show that the solution is $y = gt^2/2 + v_0 t + y_0$. How long does a fall of 100 m take if the body falls from rest? A fall of 200 m? (Guess first.)

19. (Airplane takeoff) If an airplane has a run of 3 km, starts with a speed 6 m/sec, moves with constant acceleration, and makes the run in 1 min, with what speed does it take off?

20. (Subsonic flight) The efficiency of the engines of subsonic airplanes depends on air pressure and usually is maximum near about 36 000 ft. Find the air pressure $y(x)$ at this height without calculation. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure, and at 18 000 ft the pressure has decreased to half its value y_0 at sea level.

21. (Half-life) The half-life of a radioactive substance is the time in which half of the given amount disappears. Hence it measures the rapidity of the decay. What is the half-life of radium ${}_{88}\mathrm{Ra}^{226}$ (in years) in Example 5?

22. (Interest rates) Show by algebra that the investment $y(t)$ from a deposit y_0 after t years at an interest rate r is

$$y_a(t) = y_0[1 + r]^t \qquad \text{(Interest compounded annually)}$$

$$y_d(t) = y_0[1 + (r/365)]^{365t} \qquad \text{(Interest compounded daily).}$$

Recall from calculus that

$$[1 + (1/n)]^n \to e \text{ as } n \to \infty;$$

hence $[1 + (r/n)]^{nt} \to e^{rt}$; thus

$$y_c(t) = y_0 e^{rt} \qquad \text{(Interest compounded continuously).}$$

What ODE does the last function satisfy? Let the initial investment be \$1000 and $r = 6\%$. Compute the value of the investment after 1 year and after 5 years using each of the three formulas. Is there much difference?

1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point (x_0, y_0) must have at that point the slope $y'(x_0)$ equal to the value of f at that point; that is,

$$y'(x_0) = f(x_0, y_0).$$

Read this paragraph again before you go on, and think about it.

It follows that you can indicate directions of solution curves of (1) by drawing short straight-line segments (*lineal elements*) in the xy-plane (as in Fig. 7a) and then fitting (approximate) solution curves through the direction field (or *slope field*) thus obtained. This method is important for two reasons.

1. You need not solve (1). This is essential because many ODEs have complicated solution formulas or none at all.
2. The method shows, in graphical form, the whole family of solutions and their typical properties. The accuracy is somewhat limited, but in most cases this does not matter.

Let us illustrate this method for the ODE

$$y' = xy. \tag{2}$$

Direction Fields by a CAS (Computer Algebra System). A CAS plots lineal elements at the points of a square grid, as in Fig. 7a for (2), into which you can fit solution curves. Decrease the mesh size of the grid in regions where $f(x, y)$ varies rapidly.

Direction Fields by Using Isoclines (the Older Method). Graph the curves $f(x, y) = k = const$, called *isoclines* (meaning *curves of equal inclination*). For (2) these are the hyperbolas $f(x, y) = xy = k = const$ (and the coordinate axes) in Fig. 7b. By (1), these are the curves along which the derivative y' is constant. These are not yet solution curves—don't get confused. Along each isocline draw many parallel line elements of the corresponding slope k. This gives the direction field, into which you can now graph approximate solution curves.

We mention that for the ODE (2) in Fig. 7 we would not need the method, because we shall see in the next section that ODEs such as (2) can easily be solved exactly. For the time being, let us verify by substitution that (2) has the general solution

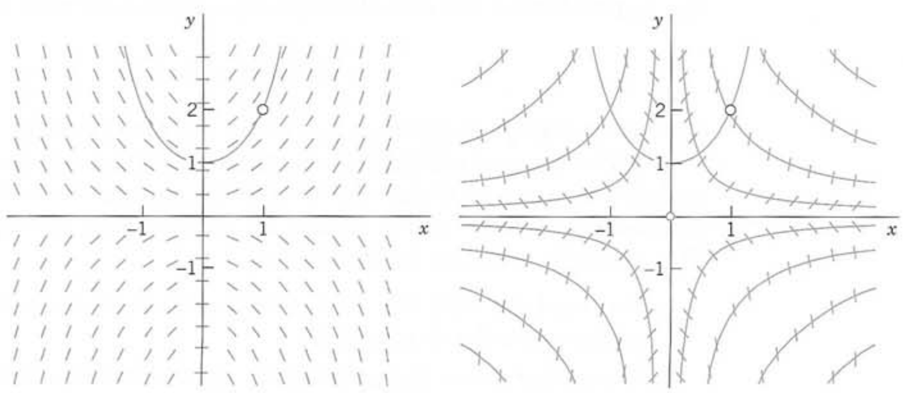
$$y(x) = ce^{x^2/2} \qquad (c \text{ arbitrary}).$$

Indeed, by differentiation (chain rule!) we get $y' = x(ce^{x^2/2}) = xy$. Of course, knowing the solution, we now have the advantage of obtaining a feel for the accuracy of the method by comparing with the exact solution. The particular solution in Fig. 7 through $(x, y) = (1, 2)$ must satisfy $y(1) = 2$. Thus, $2 = ce^{1/2}$, $c = 2/\sqrt{e} = 1.213$, and the particular solution is $y(x) = 1.213e^{x^2/2}$.

A famous ODE for which we *do* need direction fields is

$$y' = 0.1(1 - x^2) - \frac{x}{y}. \tag{3}$$

(It is related to the van der Pol equation of electronics, which we shall discuss in Sec. 4.5.) The direction field in Fig. 8 shows lineal elements generated by the computer. We have also added the isoclines for $k = -5, -3, \frac{1}{4}, 1$ as well as three typical solution curves, one that is (almost) a circle and two spirals approaching it from inside and outside.



(a) By a CAS

(b) By isoclines

Fig. 7. Direction field of $y' = xy$

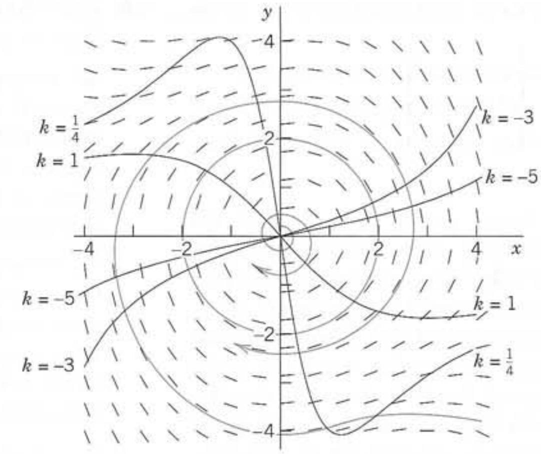


Fig. 8. Direction field of $y' = 0.1(1 - x^2) - \dfrac{x}{y}$

On Numerics

Direction fields give "all" solutions, but with limited accuracy. If we need accurate numeric values of a solution (or of several solutions) for which we have no formula, we can use a **numeric method.** If you want to get an idea of how these methods work, go to Sec. 21.1 and study the first two pages on the **Euler–Cauchy method,** which is typical of more accurate methods later in that section, notably of the classical **Runge–Kutta method.** It would make little sense to interrupt the present flow of ideas by including such methods here; indeed, it would be a duplication of the material in Sec. 21.1. For an excursion to that section you need no extra prerequisites; Sec. 1.1 just discussed is sufficient.

PROBLEM SET 1.2

1–10 DIRECTION FIELDS, SOLUTION CURVES

Graph a direction field (by a CAS or by hand). In the field graph approximate solution curves through the given point or points (x, y) by hand.

1. $y' = e^x - y$, $(0, 0)$, $(0, 1)$

2. $4yy' = -9x$, $(2, 2)$

3. $y' = 1 + y^2$, $(\frac{1}{4}\pi, 1)$

4. $y' = y - 2y^2$, $(0, 0)$, $(0, 0.25)$, $(0, 0.5)$, $(0, 1)$

5. $y' = x^2 - 1/y$, $(1, -2)$

6. $y' = 1 + \sin y$, $(-1, 0)$, $(1, -4)$

7. $y' = y^3 + x^3$, $(0, 1)$

8. $y' = 2xy + 1$, $(-1, 2)$, $(0, 0)$, $(1, -2)$

9. $y' = y \tanh x - 2$, $(-1, -2)$, $(1, 0)$, $(1, 2)$

10. $y' = e^{y/x}$, $(1, 1)$, $(2, 2)$, $(3, 3)$

11–15 ACCURACY

Direction fields are very useful because you can see solutions (as many as you want) without solving the ODE, which may be difficult or impossible in terms of a formula. To get a feel for the accuracy of the method, graph a field, sketch solution curves in it, and compare them with the exact solutions.

11. $y' = \sin \frac{1}{2}\pi x$

12. $y' = 1/x^2$

13. $y' = -2y$ (Sol. $y = ce^{-2x}$)

14. $y' = 3y/x$ (Sol. $y = cx^3$)

15. $y' = -\ln x$

16–18 MOTIONS

A body moves on a straight line, with velocity as given, and $y(t)$ is its distance from a fixed point 0 and t time. Find a model of the motion (an ODE). Graph a direction field.

In it sketch a solution curve corresponding to the given initial condition.

16. Velocity equal to the reciprocal of the distance, $y(1) = 1$

17. Product of velocity and distance equal to $-t$, $y(3) = -3$

18. Velocity plus distance equal to the square of time, $y(0) = 6$

19. **(Skydiver)** Two forces act on a parachutist, the attraction by the earth mg (m = mass of person plus equipment, $g = 9.8$ m/sec^2 the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using **Newton's second law** of motion (mass $\times$ acceleration = resultant of the forces), set up a model (an ODE for $v(t)$). Graph a direction field (choosing m and the constant of proportionality equal to 1). Assume that the parachute opens when $v = 10$ m/sec. Graph the corresponding solution in the field. What is the limiting velocity?

20. **CAS PROJECT. Direction Fields.** Discuss direction fields as follows.

(a) Graph a direction field for the ODE $y' = 1 - y$ and in it the solution satisfying $y(0) = 5$ showing **exponential approach.** Can you see the limit of any solution directly from the ODE? For what initial condition will the solution be increasing? Constant? Decreasing?

(b) What do the solution curves of $y' = -x^3/y^3$ look like, as concluded from a direction field. How do they seem to differ from circles? What are the isoclines? What happens to those curves when you drop the minus on the right? Do they look similar to familiar curves? First, guess.

(c) Compare, as best as you can, the old and the computer methods, their advantages and disadvantages. Write a short report.

1.3 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

$$g(y)y' = f(x) \qquad (1)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

$$\int g(y)\, y'\, dx = \int f(x)\, dx + c. \qquad (2)$$

On the left we can switch to y as the variable of integration. By calculus, $y'\, dx = dy$, so that

$$\int g(y)\, dy = \int f(x)\, dx + c. \qquad (3)$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables,** and (1) is called a **separable equation,** because in (3) the variables are now separated: x appears only on the right and y only on the left.

EXAMPLE 1 **A Separable ODE**

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1 + y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

It is very important to introduce the constant of integration immediately when the integration is performed. If we wrote arctan $y = x$, then $y = \tan x$, and *then* introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$). Verify this. ■

Modeling

The importance of modeling was emphasized in Sec. 1.1, and separable equations yield various useful models. Let us discuss this in terms of some typical examples.

EXAMPLE 2 Radiocarbon Dating[2]

In September 1991 the famous Iceman (Oetzi), a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps (hence the name "Oetzi") in Southern Tyrolia near the Austrian–Italian border, caused a scientific sensation. When did Oetzi approximately live and die if the ratio of carbon ${}_6C^{14}$ to carbon ${}_6C^{12}$ in this mummy is 52.5% of that of a living organism?

Physical Information. In the atmosphere and in living organisms, the ratio of radioactive carbon ${}_6C^{14}$ (made radioactive by cosmic rays) to ordinary carbon ${}_6C^{12}$ is constant. When an organism dies, its absorption of ${}_6C^{14}$ by breathing and eating terminates. Hence one can estimate the age of a fossil by comparing the radioactive carbon ratio in the fossil with that in the atmosphere. To do this, one needs to know the half-life of ${}_6C^{14}$, which is 5715 years (*CRC Handbook of Chemistry and Physics,* 83rd ed., Boca Raton: CRC Press, 2002, page 11–52, line 9).

Solution. ***Modeling.*** Radioactive decay is governed by the ODE $y' = ky$ (see Sec. 1.1, Example 5). By separation and integration (where t is time and y_0 is the initial ratio of ${}_6C^{14}$ to ${}_6C^{12}$)

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0 e^{kt}.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Oetzi died (actually, was killed),

$$e^{kt} = e^{-0.0001213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001213} = 5312. \qquad \textit{Answer:} \qquad \text{About 5300 years ago.}$$

Other methods show that radiocarbon dating values are usually too small. According to recent research, this is due to a variation in that carbon ratio because of industrial pollution and other factors, such as nuclear testing. ■

EXAMPLE 3 Mixing Problem

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 9 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{"Balance law".}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is $10/1000 = 0.01$ ($= 1\%$) of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01y(t)$. Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{4}$$

[2]Method by WILLARD FRANK LIBBY (1908–1980), American chemist, who was awarded for this work the 1960 Nobel Prize in chemistry.

Step 2. Solution of the model. The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y - 5000} = -0.01\,dt, \qquad \ln|y - 5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = -4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{5}$$

This function shows an exponential approach to the limit 5000 lb; see Fig. 9. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 27) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly. ■

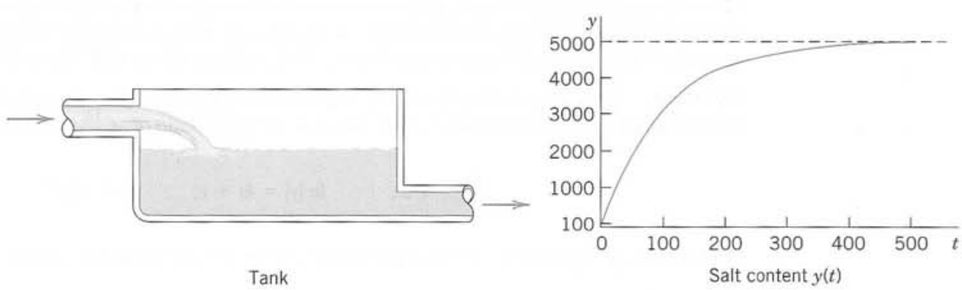


Fig. 9. Mixing problem in Example 3

EXAMPLE 4 Heating an Office Building (Newton's Law of Cooling[3])

Suppose that in Winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, as, for example, a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium **(Newton's law of cooling).**

Solution. ***Step 1. Setting up a model.*** Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A). \tag{6}$$

Such experimental laws are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly (as in the present case), it may still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with calculations from the model.

---

[3]Sir ISAAC NEWTON (1642–1727), great English physicist and mathematician, became a professor at Cambridge in 1669 and Master of the Mint in 1699. He and the German mathematician and philosopher GOTTFRIED WILHELM LEIBNIZ (1646–1716) invented (independently) the differential and integral calculus. Newton discovered many basic physical laws and created the method of investigating physical problems by means of calculus. His *Philosophiae naturalis principia mathematica* (*Mathematical Principles of Natural Philosophy*, 1687) contains the development of classical mechanics. His work is of greatest importance to both mathematics and physics.

Step 2. General solution. We cannot solve (6) because we do not know T_A, just that it varied between 50°F and 40°F, so we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one.* We solve (6) with the unknown function T_A replaced with the average of the two known values, or 45°F. For physical reasons we may expect that this will give us a reasonable approximate value of T in the building at 6 A.M.

For constant $T_A = 45$ (or any other *constant* value) the ODE (6) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T-45} = k\,dt, \qquad \ln|T-45| = kt + c^*, \qquad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*}).$$

Step 3. Particular solution. We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution,

$$T(0) = 45 + ce^{0} = 70, \qquad c = 70 - 45 = 25, \qquad T_p(t) = 45 + 25e^{kt}.$$

Step 4. Determination of k. We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (Fig. 10)

$$T_p(4) = 45 + 25e^{4k} = 65, \qquad e^{4k} = 0.8, \qquad k = \tfrac{1}{4}\ln 0.8 = -0.056, \qquad T_p(t) = 45 + 25e^{-0.056t}.$$

Step 5. Answer and interpretation. 6 A.M. is $t = 8$ (namely, 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056 \cdot 8} = 61[°\text{F}].$$

Hence the temperature in the building dropped 9°F, a result that looks reasonable. ■

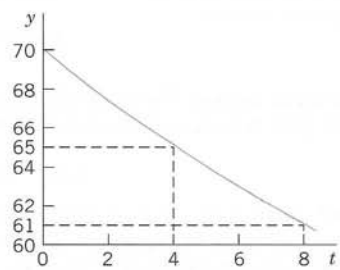


Fig. 10. Particular solution (temperature) in Example 4

EXAMPLE 5 Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 11). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Physical information. Under the influence of gravity the outflowing water has velocity

$$v(t) = 0.600\sqrt{2gh(t)} \qquad \textbf{(Torricelli's law}^4\textbf{)}, \tag{7}$$

where $h(t)$ is the height of the water above the hole at time t, and $q = 980\ \text{cm/sec}^2 = 32.17\ \text{ft/sec}^2$ is the acceleration of gravity at the surface of the earth.

Solution. ***Step 1. Setting up the model.*** To get an equation, we relate the decrease in water level $h(t)$ to the outflow. The volume ΔV of the outflow during a short time Δt is

$$\Delta V = Av\,\Delta t \qquad (A = \text{Area of hole}).$$

[4]EVANGELISTA TORRICELLI (1608–1647), Italian physicist, pupil and successor of GALILEO GALILEI (1564–1642) at Florence. The "contraction factor" 0.600 was introduced by J. C. BORDA in 1766 because the stream has a smaller cross section than the area of the hole.

ΔV must equal the change ΔV^* of the volume of the water in the tank. Now

$$\Delta V^* = -B\,\Delta h \qquad (B = \text{Cross-sectional area of tank})$$

where Δh (> 0) is the decrease of the height $h(t)$ of the water. The minus sign appears because the volume of the water in the tank decreases. Equating ΔV and ΔV^* gives

$$-B\,\Delta h = Av\,\Delta t.$$

We now express v according to Torricelli's law and then let Δt (the length of the time interval considered) approach 0—this is a ***standard way*** of obtaining an ODE as a model. That is, we have

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}\,v = -\frac{A}{B}\,0.600\sqrt{2gh(t)},$$

and by letting $\Delta t \to 0$ we obtain the ODE

$$\frac{dh}{dt} = -26.56\,\frac{A}{B}\,\sqrt{h},$$

where $26.56 = 0.600\,\sqrt{2 \cdot 980}$. This is our model, a first-order ODE.

Step 2. General solution. Our ODE is separable. A/B is constant. Separation and integration gives

$$\frac{dh}{\sqrt{h}} = -26.56\,\frac{A}{B}\,dt \qquad \text{and} \qquad 2\sqrt{h} = c^* - 26.56\,\frac{A}{B}\,t.$$

Dividing by 2 and squaring gives $h = (c - 13.28At/B)^2$. Inserting $13.28A/B = 13.28 \cdot 0.5^2\pi/100^2\pi = 0.000332$ yields the general solution

$$h(t) = (c - 0.000332t)^2.$$

Step 3. Particular solution. The initial height (the initial condition) is $h(0) = 225$ cm. Substitution of $t = 0$ and $h = 225$ gives from the general solution $c^2 = 225$, $c = 15.00$ and thus the particular solution (Fig. 11)

$$h_p(t) = (15.00 - 0.000332t)^2.$$

Step 4. Tank empty. $h_p(t) = 0$ if $t = 15.00/0.000332 = 45\,181$ [sec] $= 12.6$ [hours].

Here you see distinctly the ***importance of the choice of units***—we have been working with the Cgs system, in which time is measured in seconds! We used $g = 980$ cm/sec^2.

Step 5. Checking. Check the result. ■

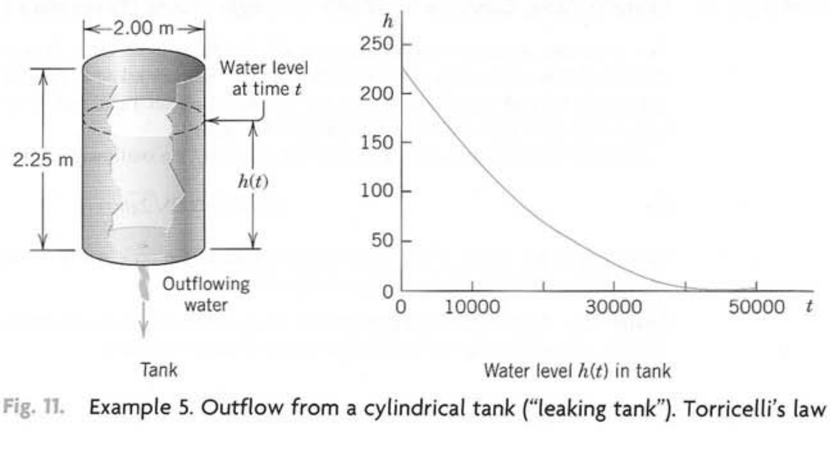


Fig. 11. Example 5. Outflow from a cylindrical tank ("leaking tank"). Torricelli's law

Extended Method: Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE,* a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

The form of such an ODE suggests that we set $y/x = u$; thus,

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u. \tag{9}$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}. \tag{10}$$

EXAMPLE 6 Reduction to Separable Form

Solve

$$2xyy' = y^2 - x^2.$$

Solution. To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' from (9) and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2 to obtain (Fig. 12)

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis. ■

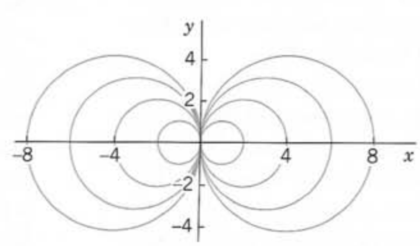


Fig. 12. General solution (family of circles) in Example 6

PROBLEM SET 1.3

1. (Constant of integration) An arbitrary constant of integration must be introduced immediately when the integration is performed. Why is this important? Give an example of your own.

2–9 GENERAL SOLUTION

Find a general solution. Show the steps of derivation. Check your answer by substitution.

2. $y' + (x + 2)y^2 = 0$

3. $y' = 2 \sec 2y$

4. $y' = (y + 9x)^2 \quad (y + 9x = v)$

5. $yy' + 36x = 0$

6. $y' = (4x^2 + y^2)/(xy)$

7. $y' \sin \pi x = y \cos \pi x$

8. $xy' = \frac{1}{2}y^2 + y$

9. $y'e^{\pi x} = y^2 + 1$

10–19 INITIAL VALUE PROBLEMS

Find the particular solution. Show the steps of derivation, beginning with the general solution. (L, R, b are constants.)

10. $yy' + 4x = 0,\ y(0) = 3$

11. $dr/dt = -2tr,\ r(0) = r_0$

12. $2xyy' = 3y^2 + x^2,\ y(1) = 2$

13. $L\, dI/dt + RI = 0,\ I(0) = I_0$

14. $y' = y/x + (2x^3/y)\cos(x^2),\ y(\sqrt{\pi/2}) = \sqrt{\pi}$

15. $e^{2x}y' = 2(x + 2)y^3,\ y(0) = 1/\sqrt{5} \approx 0.45$

16. $xy' = y + 4x^5 \cos^2(y/x),\ y(2) = 0$

17. $y'x \ln x = y,\ y(3) = \ln 81$

18. $dr/d\theta = b[(dr/d\theta) \cos \theta + r \sin \theta],\ r(\frac{1}{2}\pi) = \pi,\ 0 < b < 1$

19. $yy' = (x - 1)e^{-y^2},\ y(0) = 1$

20. (Particular solution) Introduce limits of integration in (3) such that y obtained from (3) satisfies the initial condition $y(x_0) = y_0$. Try the formula out on Prob. 19.

21–36 APPLICATIONS, MODELING

21. (Curves) Find all curves in the xy-plane whose tangents all pass through a given point (a, b).

22. (Curves) Show that any (nonvertical) straight line through the origin of the xy-plane intersects all solution curves of $y' = g(y/x)$ at the same angle.

23. (Exponential growth) If the growth rate of the amount of yeast at any time t is proportional to the amount present at that time and doubles in 1 week, how much yeast can be expected after 2 weeks? After 4 weeks?

24. (Population model) If in a population of bacteria the birth rate and death rate are proportional to the number of individuals present, what is the population as a function of time? Figure out the limiting situation for increasing time and interpret it.

25. (Radiocarbon dating) If a fossilized tree is claimed to be 4000 years old, what should be its ${}_6C^{14}$ content expressed as a percent of the ratio of ${}_6C^{14}$ to ${}_6C^{12}$ in a living organism?

26. (Gompertz growth in tumors) The Gompertz model is $y' = -Ay \ln y$ ($A > 0$), where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.

27. (Dryer) If wet laundry loses half of its moisture during the first 5 minutes of drying in a dryer and if the rate of loss of moisture is proportional to the moisture content, when will the laundry be practically dry, say, when will it have lost 95% of its moisture? First guess.

28. (Alibi?) Jack, arrested when leaving a bar, claims that he has been inside for at least half an hour (which would provide him with an alibi). The police check the water temperature of his car (parked near the entrance of the bar) at the instant of arrest and again 30 minutes later, obtaining the values 190°F and 110°F, respectively. Do these results give Jack an alibi? (Solve by inspection.)

29. (Law of cooling) A thermometer, reading 10°C, is brought into a room whose temperature is 23°C. Two minutes later the thermometer reading is 18°C. How long will it take until the reading is practically 23°C, say, 22.8°C? First guess.

30. (Torricelli's law) How does the answer in Example 5 (the time when the tank is empty) change if the diameter of the hole is doubled? First guess.

31. (Torricelli's law) Show that (7) looks reasonable inasmuch as $\sqrt{2gh(t)}$ is the speed a body gains if it falls a distance h (and air resistance is neglected).

32. (Rope) To tie a boat in a harbor, how many times must a rope be wound around a bollard (a vertical rough cylindrical post fixed on the ground) so that a man holding one end of the rope can resist a force exerted by the boat one thousand times greater than the man can exert? First guess. Experiments show that the change ΔS of the force S in a small portion of the rope is proportional to S and to the small angle $\Delta\phi$ in Fig. 13. Take the proportionality constant 0.15.

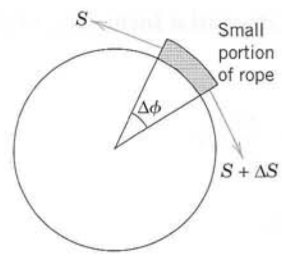


Fig. 13. Problem 32

33. **(Mixing)** A tank contains 800 gal of water in which 200 lb of salt is dissolved. Two gallons of fresh water runs in per minute, and 2 gal of the mixture in the tank, kept uniform by stirring, runs out per minute. How much salt is left in the tank after 5 hours?

34. **WRITING PROJECT. Exponential Increase, Decay, Approach.** Collect, order, and present all the information on the ODE $y' = ky$ and its applications from the text and the problems. Add examples of your own.

35. **CAS EXPERIMENT. Graphing Solutions.** A CAS can usually graph solutions even if they are given by integrals that cannot be evaluated by the usual methods of calculus. Show this as follows.

(A) Graph the curves for the seven initial value problems $y' = e^{-x^2/2}$, $y(0) = 0, \pm 1, \pm 2, \pm 3$, common axes. Are these curves congruent? Why?

(B) Experiment with approximate curves of nth partial sums of the Maclaurin series obtained by termwise integration of that of y in (A); graph them and describe qualitatively the accuracy for a fixed interval $0 \leqq x \leqq b$ and increasing n, and then for fixed n and increasing b.

(C) Experiment with $y' = \cos(x^2)$ as in (B).

(D) Find an initial value problem with solution $y = e^{x^2} \int_0^x e^{-t^2}\, dt$ and experiment with it as in (B).

36. **TEAM PROJECT. Torricelli's Law.** Suppose that the tank in Example 5 is hemispherical, of radius R, initially full of water, and has an outlet of 5 cm^2 cross-sectional area at the bottom. (Make a sketch.) Set up the model for outflow. Indicate what portion of your work in Example 5 you can use (so that it can become part of the general method independent of the shape of the tank). Find the time t to empty the tank (a) for any R, (b) for $R = 1$ m. Plot t as function of R. Find the time when $h = R/2$ (a) for any R, (b) for $R = 1$ m.

1.4 Exact ODEs. Integrating Factors

We remember from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy.$$

From this it follows that if $u(x, y) = c = const$, then $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)\, dx + 3x^2y^2\, dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2},$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'\, dx$ as in Sec. 1.3)

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact,** that is, this form is the differential

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{2}$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

$$u(x, y) = c. \tag{3}$$

This is called an **implicit solution,** in contrast with a solution $y = h(x)$ as defined in Sec. 1.1, which is also called an *explicit solution,* for distinction. Sometimes an implicit solution can be converted to explicit form. (Do this for $x^2 + y^2 = 1$.) If this is not possible, your CAS may graph a figure of the **contour lines** (3) of the function $u(x, y)$ and help you in understanding the solution.

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

$$\text{(a)}\quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)}\quad \frac{\partial u}{\partial y} = N. \tag{4}$$

From this we can derive a formula for checking whether (1) is exact or not, as follows.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections. Then by partial differentiation of (4) (see App. 3.2 for notation),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$
$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{5}$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation. (We shall prove this in Sec. 10.2 in another context. Some calculus books (e.g., Ref. [GR11] also contain a proof.)

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

(6) $$u = \int M\,dx + k(y);$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k.

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use (4b). Then instead of (6) we first have by integration with respect to y

(6*) $$u = \int N\,dy + l(x).$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

EXAMPLE 1 An Exact ODE

Solve

(7) $$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0.$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

(8) $$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y).$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

(9) $$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \cos(x + y)\,dx + (\cos(x + y) + 3y^2 + 2y)\,dy = 0.$$

This completes the check. ■

EXAMPLE 2 An Initial Value Problem

Solve the initial value problem

$$(\cos y \sinh x + 1)\, dx - \sin y \cosh x\, dy = 0, \qquad y(1) = 2. \tag{10}$$

Solution. You may verify that the given ODE is exact. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x\, dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = M = \cos y \sinh x + 1$. Hence $dl/dx = 1$. By integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$. Hence the answer is $\cos y \cosh x + x = 0.358$. Figure 14 shows the particular solutions for $c = 0, 0.358$ (thicker curve), 1, 2, 3. Check that the answer satisfies the ODE. (Proceed as in Example 1.) Also check that the initial condition is satisfied. ■

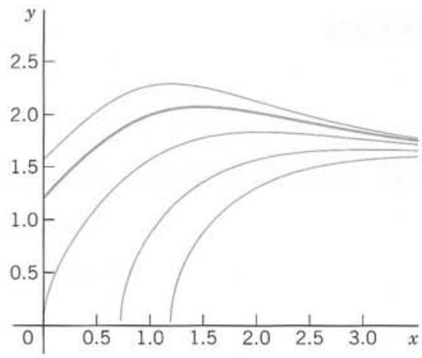


Fig. 14. Particular solu.ions in Example 2

EXAMPLE 3 WARNING! Breakdown in the Case of Nonexactness

The equation $-y\, dx + x\, dy = 0$ is not exact because $M = -y$ and $N = x$, so that in (5), $\partial M/\partial y = -1$ but $\partial N/\partial x = 1$. Let us show that in such a case the present method does not work. From (6),

$$u = \int M\, dx + k(y) = -xy + k(y), \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed. ■

Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\, dx + x\, dy = 0$. It is not exact. However, if we multiply it by $1/x^2$, we get an exact equation [check exactness by (5)!],

$$\frac{-y\, dx + x\, dy}{x^2} = -\frac{y}{x^2}\, dx + \frac{1}{x}\, dy = d\left(\frac{y}{x}\right) = 0. \tag{11}$$

Integration of (11) then gives the general solution $y/x = c = const.$

This example gives the idea. All we did was multiply a given nonexact equation, say,

$$P(x, y)\,dx + Q(x, y)\,dy = 0, \tag{12}$$

by a function F that, in general, will be a function of both x and y. The result was an equation

$$FP\,dx + FQ\,dy = 0 \tag{13}$$

that is exact, so we can solve it as just discussed. Such a function $F(x, y)$ is then called an **integrating factor** of (12).

EXAMPLE 4 Integrating Factor

The integrating factor in (11) is $F = 1/x^2$. Hence in this case the exact equation (13) is

$$FP\,dx + FQ\,dy = \frac{-y\,dx + x\,dy}{x^2} = d\left(\frac{y}{x}\right) = 0. \qquad \text{Solution} \qquad \frac{y}{x} = c.$$

These are straight lines $y = cx$ through the origin.

It is remarkable that we can readily find other integrating factors for the equation $-y\,dx + x\,dy = 0$, namely, $1/y^2$, $1/(xy)$, and $1/(x^2 + y^2)$, because

$$\frac{-y\,dx + x\,dy}{y^2} = d\left(\frac{x}{y}\right), \quad \frac{-y\,dx + x\,dy}{xy} = -d\left(\ln\frac{x}{y}\right), \quad \frac{-y\,dx + x\,dy}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right). \tag{14}$$ ■

How to Find Integrating Factors

In simpler cases we may find integrating factors by inspection or perhaps after some trials, keeping (14) in mind. In the general case, the idea is the following.

For $M\,dx + N\,dy = 0$ the exactness condition (4) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FP\,dx + FQ\,dy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \tag{15}$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_yP + FP_y = F_xQ + FQ_x.$$

In the general case, this would be complicated and useless. So we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one*—the result may be useful (and may also help you later on). Hence we look for an integrating factor depending only on ***one*** variable; fortunately, in many practical cases, there are such factors, as we shall see. Thus, let $F = F(x)$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (15) becomes

$$FP_y = F'Q + FQ_x.$$

Dividing by FQ and reshuffling terms, we have

$$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right). \tag{16}$$

This proves the following theorem.

THEOREM 1

Integrating Factor $F(x)$

If (12) is such that the right side R of (16), depends only on x, then (12) has an integrating factor $F = F(x)$, which is obtained by integrating (16) and taking exponents on both sides,

$$F(x) = \exp \int R(x)\, dx. \tag{17}$$

Similarly, if $F^* = F^*(y)$, then instead of (16) we get

$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{18}$$

and we have the companion

THEOREM 2

Integrating Factor $F^*(y)$

If (12) is such that the right side R^* of (18) depends only on y, then (12) has an integrating factor $F^* = F^*(y)$, which is obtained from (18) in the form

$$F^*(y) = \exp \int R^*(y)\, dy. \tag{19}$$

EXAMPLE 5 **Application of Theorems 1 and 2. Initial Value Problem**

Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

$$(e^{x+y} + ye^y)\, dx + (xe^y - 1)\, dy = 0, \quad y(0) = -1 \tag{20}$$

Solution. ***Step 1. Nonexactness.*** The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^y) = e^{x+y} + e^y + ye^y \qquad \text{but} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y - 1) = e^y.$$

Step 2. Integrating factor. General solution. Theorem 1 fails because R [the right side of (16)] depends on both x and y,

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (18) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1.$$

Hence (19) gives the integrating factor $F^*(y) = e^{-y}$. From this result and (20) you get the exact equation

$$(e^x + y)\, dx + (x - e^{-y})\, dy = 0.$$

Test for exactness; you will get 1 on both sides of the exactness condition. By integration, using (4a),

$$u = \int (e^x + y)\, dx = e^x + xy + k(y).$$

Differentiate this with respect to y and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

Step 3. Particular solution. The initial condition $y(0) = 1$ gives $u(0, -1) = 1 + 0 + e = 3.72$. Hence the answer is $e^x + xy + e^{-y} = 1 + e = 3.72$. Figure 15 shows several particular solutions obtained as level curves of $u(x, y) = c$, obtained by a CAS, a convenient way in cases in which it is impossible or difficult to cast a solution into explicit form. Note the curve that (nearly) satisfies the initial condition.

Step 4. Checking. Check by substitution that the answer satisfies the given equation as well as the initial condition. ■

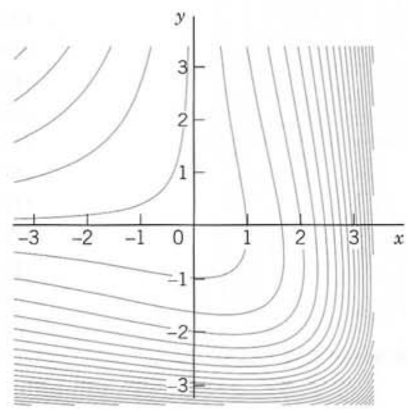


Fig. 15. Particular solutions in Example 5

PROBLEM SET 1.4

1–20 EXACT ODEs. INTEGRATING FACTORS

Test for exactness. If exact, solve. If not, use an integrating factor as given or find it by inspection or from the theorems in the text. Also, if an initial condition is given, determine the corresponding particular solution.

1. $x^3\, dx + y^3\, dy = 0$ **2.** $(x - y)(dx - dy) = 0$

3. $-\pi \sin \pi x \sinh y\, dx + \cos \pi x \cosh y\, dy = 0$

4. $(e^y - ye^x)\, dx + (xe^y - e^x)\, dy = 0$

5. $9x\, dx + 4y\, dy = 0$

6. $e^x(\cos y\, dx - \sin y\, dy) = 0$

7. $e^{-2\theta}\, dr - 2re^{-2\theta}\, d\theta = 0$

8. $(2x + 1/y - y/x^2)\, dx + (2y + 1/x - x/y^2)\, dy = 0$

9. $(-y/x^2 + 2 \cos 2x)\, dx + (1/x - 2 \sin 2y)\, dy = 0$

10. $-2xy \sin (x^2)\, dx + \cos (x^2)\, dy = 0$

11. $-y\, dx + x\, dy = 0$

12. $(e^{x+y} - y)\, dx + (xe^{x+y} + 1)\, dy = 0$

13. $-3y\, dx + 2x\, dy = 0, \quad F(x, y) = y/x^4$

14. $(x^4 + y^2)\, dx - xy\, dy = 0, \quad y(2) = 1$

15. $e^{2x}(2 \cos y\, dx - \sin y\, dy) = 0, \quad y(0) = 0$

16. $-\sin xy\, (y\, dx + x\, dy) = 0, \quad y(1) = \pi$

17. $(\cos \omega x + \omega \sin \omega x)\, dx + e^x\, dy = 0, y(0) = 1$

18. $(\cos xy + x/y)\, dx + (1 + (x/y) \cos xy)\, dy = 0$

19. $e^{-y}\, dx + e^{-x}(-e^{-y} + 1)\, dy = 0, \quad F = e^{x+y}$

20. $(\sin y \cos y + x \cos^2 y)\, dx + x\, dy = 0$

21. Under what conditions for the constants A, B, C, D is $(Ax + By)\, dx + (Cx + Dy)\, dy = 0$ exact? Solve the exact equation.

22. CAS PROJECT. Graphing Particular Solutions Graph particular solutions of the following ODE, proceeding as explained.

$$y \cos x \, dx + \frac{1}{y} \, dy = 0 \qquad (21)$$

(a) Test for exactness. If necessary, find an integrating factor. Find the general solution $u(x, y) = c$.

(b) Solve (21) by separating variables. Is this simpler than **(a)**?

(c) Graph contours $u(x, y) = c$ by your CAS. (Cf. Fig. 16.)

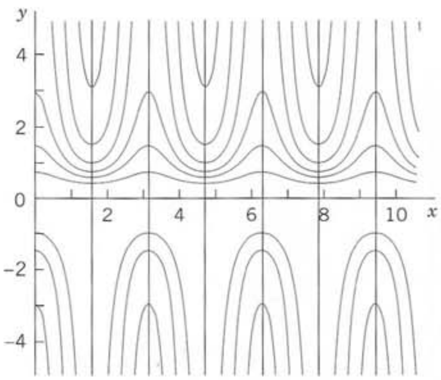


Fig. 16. Particular solutions in CAS Project 22

(d) In another graph show the solution curves satisfying $y(0) = \pm 1, \pm 2, \pm 3, \pm 4$. Compare the quality of **(c)** and **(d)** and comment.

(e) Do the same steps for another nonexact ODE of your choice.

23. WRITING PROJECT. Working Backward. Start from solutions $u(x, y) = c$ of your choice, find a corresponding exact ODE, destroy exactness by a multiplication or division. This should give you a feel for the form of ODEs you can reach by the method of integrating factors. (Working backward is useful in other areas, too; Euler and other great masters frequently did it.)

24. TEAM PROJECT. Solution by Several Methods. Show this as indicated. Compare the amount of work.

(A) $e^y(\sinh x \, dx + \cosh x \, dy) = 0$ as an exact ODE and by separation.

(B) $(1 + 2x) \cos y \, dx + dy/\cos y = 0$ by Theorem 2 and by separation.

(C) $(x^2 + y^2) \, dx - 2xy \, dy = 0$ by Theorem 1 or 2 and by separation with $v = y/x$.

(D) $3x^2 y \, dx + 4x^3 \, dy = 0$ by Theorems 1 and 2 and by separation.

(E) Search the text and the problems for further ODEs that can be solved by more than one of the methods discussed so far. Make a list of these ODEs. Find further cases of your own.

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

Linear ODEs or ODEs that can be transformed to linear form are models of various phenomena, for instance, in physics, biology, population dynamics, and ecology, as we shall see. A first-order ODE is said to be ***linear*** if it can be written

$$y' + p(x)y = r(x). \qquad (1)$$

The defining feature of this equation is that it is linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be ***any*** given functions of x. If in an application the independent variable is time, we write t instead of x.

If the first term is $f(x)y'$ (instead of y'), divide the equation by $f(x)$ to get the **"standard form"** (1), with y' as the first term, which is practical.

For instance, $y' \cos x + y \sin x = x$ is a linear ODE, and its standard form is $y' + y \tan x = x \sec x$.

The function $r(x)$ on the right may be a force, and the solution $y(x)$ a displacement in a motion or an electrical current or some other physical quantity. In engineering, $r(x)$ is frequently called the **input,** and $y(x)$ is called the **output** or the *response* to the input (and, if given, to the initial condition).

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \tag{2}$$

and is called **homogeneous.** By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\, dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\, dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)\, dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \tag{3}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous.** It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the last section. For this purpose we write (1) as

$$(py - r)\, dx + dy = 0.$$

This is $P\, dx + Q\, dy = 0$, where $P = py - r$ and $Q = 1$. Hence the right side of (16) in Sec. 1.4 is simply $1(p - 0) = p$, so that (16) becomes

$$\frac{1}{F}\frac{dF}{dx} = p(x).$$

Separation and integration gives

$$\frac{dF}{F} = p\, dx \qquad \text{and} \qquad \ln|F| = \int p\, dx.$$

Taking exponents on both sides, we obtain the desired integrating factor $F(x)$,

$$F(x) = e^{\int p\, dx}.$$

We now multiply (1) on both sides by this F. Then by the product rule,

$$e^{\int p\, dx}(y' + py) = (e^{\int p\, dx}y)' = e^{\int p\, dx}r.$$

By integrating the second and third of these three expressions with respect to x we get

$$e^{\int p\, dx}y = \int e^{\int p\, dx} r\, dx + c.$$

Dividing this equation by $e^{\int p\, dx}$ and denoting the exponent $\int p\, dx$ by h, we obtain

$$y(x) = e^{-h}\left(\int e^h r\, dx + c\right), \qquad h = \int p(x)\, dx. \tag{4}$$

(The constant of integration in h does not matter; see Prob. 2.) Formula (4) is the general solution of (1) in the form of an integral. Solving (1) is now reduced to the evaluation of an integral. In cases in which this cannot be done by the usual methods of calculus, one may have to use a numeric method for integrals (Sec. 19.5) or for the ODE itself (Sec. 21.1).

The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

$$y(x) = e^{-h} \int e^{h} r \, dx + ce^{-h}, \tag{4*}$$

we see the following:

(5) Total Output = Response to the Input r + Response to the Initial Data.

EXAMPLE 1 First-Order ODE, General Solution

Solve the linear ODE

$$y' - y = e^{2x}.$$

Solution. Here,

$$p = -1, \qquad r = e^{2x}, \qquad h = \int p \, dx = -x$$

and from (4) we obtain the general solution

$$y(x) = e^{x} \left(\int e^{-x} e^{2x} \, dx + c \right) = e^{x}(e^{x} + c) = ce^{x} + e^{2x}.$$

From (4*) and (5) we see that the response to the input is e^{2x}.

In simpler cases, such as the present, we may not need the general formula (4), but may wish to proceed directly, multiplying the given equation by $e^{h} = e^{-x}$. This gives

$$(y' - y)e^{-x} = (ye^{-x})' = e^{2x}e^{-x} = e^{x}.$$

Integrating on both sides, we obtain the same result as before:

$$ye^{-x} = e^{x} + c, \qquad \text{hence} \qquad y = e^{2x} + ce^{x}.$$ ■

EXAMPLE 2 First-Order ODE, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$\int p \, dx = \int \tan x \, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h} r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x \, dx + c \right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$. ■

EXAMPLE 3 Hormone Level

Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ be the hormone level at time t. Then the removal rate is $Ky(t)$. The input rate is $A + B\cos(2\pi t/24)$, where A is the average input rate, and $A \geqq B$ to make the input nonnegative. (The constants A, B, and K can be determined by measurements.) Hence the model is

$$y'(t) = \text{In} - \text{Out} = A + B\cos\left(\tfrac{1}{12}\pi t\right) - Ky(t) \qquad \text{or} \qquad y' + Ky = A + B\cos\left(\tfrac{1}{12}\pi t\right).$$

The initial condition for a particular solution y_{part} is $y_{part}(0) = y_0$ with $t = 0$ suitably chosen, e.g., 6:00 A.M.

Step 2. General solution. In (4) we have $p = K = const$, $h = Kt$, and $r = A + B\cos\left(\frac{1}{12}\pi t\right)$. Hence (4) gives the general solution

$$y(t) = e^{-Kt}\int e^{Kt}\left(A + B\cos\frac{\pi t}{12}\right)dt + ce^{-Kt}$$

$$= e^{-Kt}e^{Kt}\left[\frac{A}{K} + \frac{B}{144K^2 + \pi^2}\left(144K\cos\frac{\pi t}{12} + 12\pi\sin\frac{\pi t}{12}\right)\right] + ce^{-Kt}$$

$$= \frac{A}{K} + \frac{B}{144K^2 + \pi^2}\left(144K\cos\frac{\pi t}{12} + 12\pi\sin\frac{\pi t}{12}\right) + ce^{-Kt}.$$

The last term decreases to 0 as t increases, practically after a short time and regardless of c (that is, of the initial condition). The other part of $y(t)$ is called the **steady-state solution** because it consists of constant and periodic terms. The entire solution is called the **transient-state solution** because it models the transition from rest to the steady state. These terms are used quite generally for physical and other systems whose behavior depends on time.

Step 3. Particular solution. Setting $t = 0$ in $y(t)$ and choosing $y_0 = 0$, we have

$$y(0) = \frac{A}{K} + \frac{B}{144K^2 + \pi^2}\cdot 144K + c = 0, \qquad \text{thus} \qquad c = -\frac{A}{K} - \frac{B}{144K^2 + \pi^2}\cdot 144K.$$

Inserting this result into $y(t)$, we obtain the particular solution

$$y_{part}(t) = \frac{A}{K} + \frac{B}{144K^2 + \pi^2}\left(144K\cos\frac{\pi t}{12} + 12\pi\sin\frac{\pi t}{12}\right) - \left(\frac{A}{K} + \frac{144KB}{144K^2 + \pi^2}\right)e^{-Kt}$$

with the steady-state part as before. To plot y_{part} we must specify values for the constants, say, $A = B = 1$ and $K = 0.05$. Figure 17 shows this solution. Notice that the transition period is relatively short (although K is small), and the curve soon looks sinusoidal; this is the response to the input $A + B\cos\left(\frac{1}{12}\pi t\right) = 1 + \cos\left(\frac{1}{12}\pi t\right)$. ■

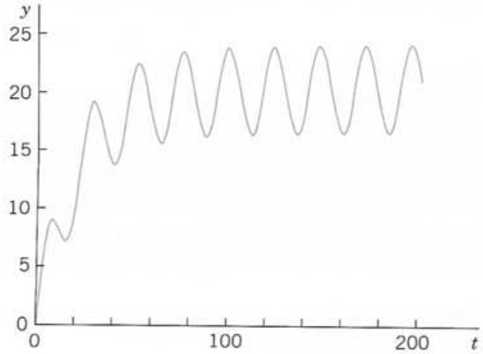


Fig. 17. Particular solution in Example 3

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**[5]

$$y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}). \tag{6}$$

If $a = 0$ or $a = 1$, Equation (6) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (6), obtaining

$$u' = (1 - a)y^{-a}y' = (1 - a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1 - a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

$$u' + (1 - a)pu = (1 - a)g. \tag{7}$$

For further ODEs reducible to linear from, see Ince's classic [A11] listed in App. 1. See also Team Project 44 in Problem Set 1.5.

EXAMPLE 4 Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**[6]):

$$y' = Ay - By^2 \tag{8}$$

Solution. Write (8) in the form (6), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$, so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (8),

$$u' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}.$$

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

---

[5] JAKOB BERNOULLI (1654–1705), Swiss mathematician, professor at Basel, also known for his contribution to elasticity theory and mathematical probability. The method for solving Bernoulli's equation was discovered by the Leibniz in 1696. Jakob Bernoulli's students included his nephew NIKLAUS BERNOULLI (1687–1759), who contributed to probability theory and infinite series, and his youngest brother JOHANN BERNOULLI (1667–1748), who had profound influence on the development of calculus, became Jakob's successor at Basel, and had among his students GABRIEL CRAMER (see Sec. 7.7) and LEONHARD EULER (see Sec. 2.5). His son DANIEL BERNOULLI (1700–1782) is known for his basic work in fluid flow and the kinetic theory of gases.

[6] PIERRE-FRANÇOIS VERHULST, Belgian statistician, who introduced Eq. (8) as a model for human population growth in 1838.

$$u' + Au = B.$$

The general solution is [by (4)]

$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (8),

$$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A} \qquad \text{(Fig. 18).} \tag{9}$$

Directly from (8) we see that $y \equiv 0$ ($y(t) = 0$ for all t) is also a solution. ■

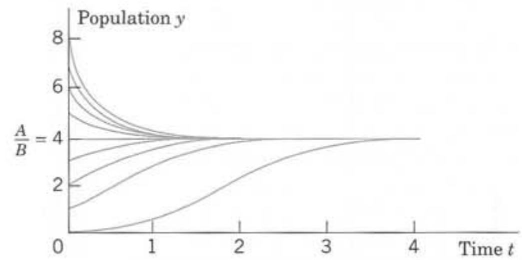


Fig. 18. Logistic population model. Curves (9) in Example 4 with $A/B = 4$

Population Dynamics

The logistic equation (8) plays an important role in **population dynamics,** a field that models the evolution of populations of plants, animals, or humans over time t. If $B = 0$, then (8) is $y' = dy/dt = Ay$. In this case its solution (9) is $y = (1/c)e^{At}$ and gives exponential growth, as for a small population in a large country (the United States in early times!). This is called ***Malthus's law.*** (See also Example 3 in Sec. 1.1.)

The term $-By^2$ in (8) is a "braking term" that prevents the population from growing without bound. Indeed, if we write $y' = Ay[1 - (B/A)y]$, we see that if $y < A/B$, then $y' > 0$, so that an initially small population keeps growing as long as $y < A/B$. But if $y > A/B$, then $y' < 0$ and the population is decreasing as long as $y > A/B$. The limit is the same in both cases, namely, A/B. See Fig. 18.

We see that in the logistic equation (8) the independent variable t does not occur explicitly. An ODE $y' = f(t, y)$ in which t does not occur explicitly is of the form

$$y' = f(y) \tag{10}$$

and is called an **autonomous ODE.** Thus the logistic equation (8) is autonomous.

Equation (10) has constant solutions, called **equilibrium solutions** or **equilibrium points.** These are determined by the zeros of $f(y)$, because $f(y) = 0$ gives $y' = 0$ by (10); hence $y = const.$ These zeros are known as **critical points** of (10). An equilibrium solution is called **stable** if solutions close to it for some t remain close to it for all further t. It is called **unstable** if solutions initially close to it do not remain close to it as t increases. For instance, $y = 0$ in Fig. 18 is an unstable equilibrium solution, and $y = 4$ is a stable one.

EXAMPLE 5 Stable and Unstable Equilibrium Solutions. "Phase Line Plot"

The ODE $y' = (y - 1)(y - 2)$ has the stable equilibrium solution $y_1 = 1$ and the unstable $y_2 = 2$, as the direction field in Fig. 19 suggests. The values y_1 and y_2 are the zeros of the parabola $f(y) = (y - 1)(y - 2)$ in the figure. Now, since the ODE is autonomous, we can "condense" the direction field to a "phase line plot" giving y_1 and y_2, and the direction (upward or downward) of the arrows in the field, and thus giving information about the stability or instability of the equilibrium solutions. ■

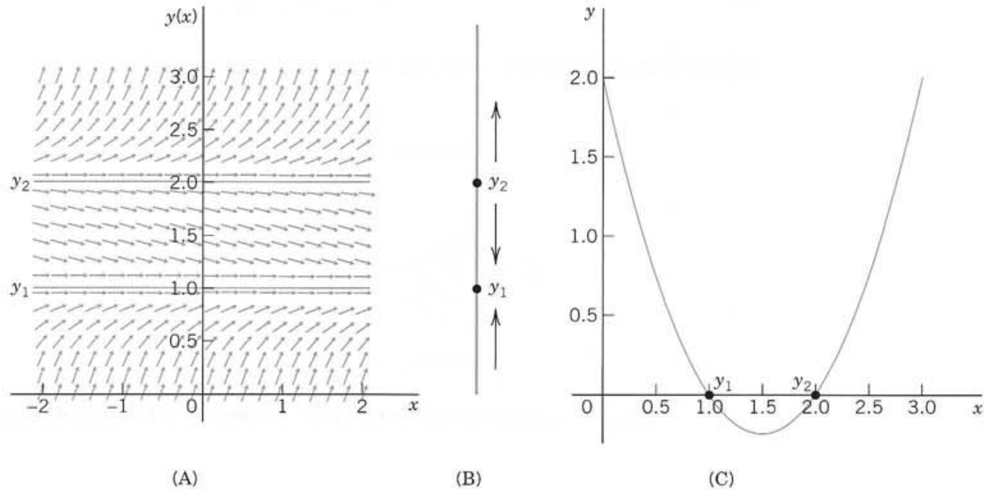


Fig. 19. Example 5. (A) Direction field. (B) "Phase line". (C) Parabola $f(y)$

A few further population models will be discussed in the problem set. For some more details of population dynamics, see C. W. Clark, *Mathematical Bioeconomics*, New York, Wiley, 1976.

Further important applications of linear ODEs follow in the next section.

PROBLEM SET 1.5

1. (CAUTION!) Show that $e^{-\ln x} = 1/x$ (not $-x$) and $e^{-\ln(\sec x)} = \cos x$.
2. **(Integration constant)** Give a reason why in (4) you may choose the constant of integration in $\int p\, dx$ to be zero.

3–17 GENERAL SOLUTION. INITIAL VALUE PROBLEMS

Find the general solution. If an initial condition is given, find also the corresponding particular solution and graph or sketch it. (Show the details of your work.)

3. $y' + 3.5y = 2.8$
4. $y' = 4y + x$
5. $y' + 1.25y = 5, \quad y(0) = 6.6$
6. $x^2y' + 3xy = 1/x, \quad y(1) = -1$
7. $y' + ky = e^{2kx}$
8. $y' + 2y = 4 \cos 2x, \quad y(\frac{1}{4}\pi) = 2$
9. $y' = 6(y - 2.5) \tanh 1.5x$
10. $y' + 4x^2y = (4x^2 - x)e^{-x^2/2}$
11. $y' + 2y \sin 2x = 2e^{\cos 2x}, \quad y(0) = 0$
12. $y' \tan x = 2y - 8, \quad y(\frac{1}{2}\pi) = 0$
13. $y' + 4y \cot 2x = 6 \cos 2x, \quad y(\frac{1}{4}\pi) = 2$
14. $y' + y \tan x = e^{-0.01x} \cos x, \quad y(0) = 0$
15. $y' + y/x^2 = 2xe^{1/x}, \quad y(1) = 13.86$
16. $y' \cos^2 x + 3y = 1, \quad y(\frac{1}{4}\pi) = \frac{4}{3}$
17. $x^3y' + 3x^2y = 5 \sinh 10x$

18–24 NONLINEAR ODEs

Using a method of this section or separating variables, find the general solution. If an initial condition is given, find also the particular solution and sketch or graph it.

18. $y' + y = y^2, \quad y(0) = -1$

19. $y' = 5.7y - 6.5y^2$

20. $(x^2 + 1)y' = -\tan y, \quad y(0) = \frac{1}{2}\pi$

21. $y' + (x + 1)y = e^{x^2}y^3, \quad y(0) = 0.5$

22. $y' \sin 2y + x \cos 2y = 2x$

23. $2yy' + y^2 \sin x = \sin x, \quad y(0) = \sqrt{2}$

24. $y' + x^2 y = (e^{-x^3} \sinh x)/(3y^2)$

25–36 FURTHER APPLICATIONS

25. (Investment programs) Bill opens a retirement savings account with an initial amount y_0 and then adds \$$k$ to the account at the beginning of every year until retirement at age 65. Assume that the interest is compounded continuously at the same rate R over the years. Set up a model for the balance in the account and find the general solution as well as the particular solution, letting $t = 0$ be the instant when the account is opened. How much money will Bill have in the account at age 65 if he starts at 25 and invests \$1000 initially as well as annually, and the interest rate R is 6%? How much should he invest initially and annually (same amounts) to obtain the same final balance as before if he starts at age 45? First, guess.

26. (Mixing problem) A tank (as in Fig. 9 in Sec. 1.3) contains 1000 gal of water in which 200 lb of salt is dissolved. 50 gal of brine, each gallon containing $(1 + \cos t)$ lb of dissolved salt, runs into the tank per minute. The mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t (Fig. 20).

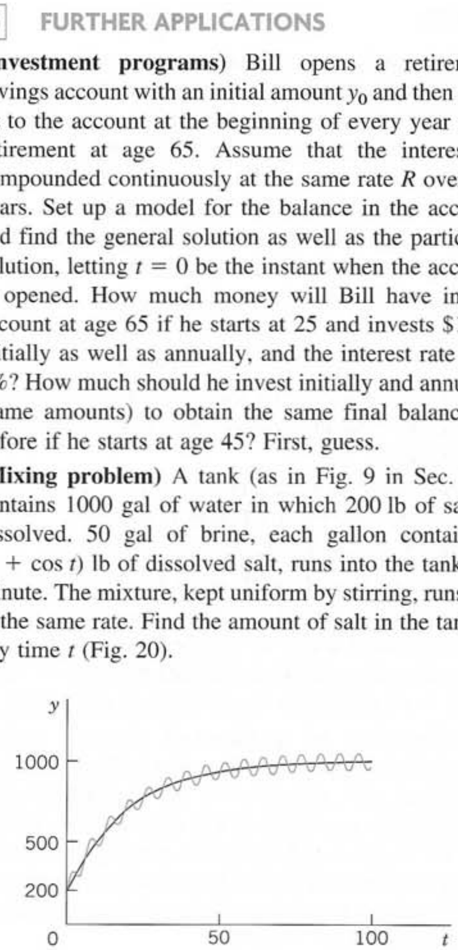


Fig. 20. Amount of salt $y(t)$ in the tank in Problem 26

27. (Lake Erie) Lake Erie has a water volume of about 450 km³ and a flow rate (in and out) of about 175 km³ per year. If at some instant the lake has pollution concentration $p = 0.04\%$, how long, approximately, will it take to decrease it to $p/2$, assuming that the inflow is much cleaner, say, it has pollution concentration $p/4$, and the mixture is uniform (an assumption that is only very imperfectly true)? First, guess.

28. (Heating and cooling of a building) Heating and cooling of a building can be modeled by the ODE

$$T' = k_1(T - T_a) + k_2(T - T_w) + P,$$

where $T = T(t)$ is the temperature in the building at time t, T_a the outside temperature, T_w the temperature wanted in the building, and P the rate of increase of T due to machines and people in the building, and k_1 and k_2 are (negative) constants. Solve this ODE, assuming $P = const$, $T_w = const$, and T_a varying sinusoidally over 24 hours, say, $T_a = A - C \cos(2\pi/24)t$. Discuss the effect of each term of the equation on the solution.

29. (Drug injection) Find and solve the model for drug injection into the bloodstream if, beginning at $t = 0$, a constant amount A g/min is injected and the drug is simultaneously removed at a rate proportional to the amount of the drug present at time t.

30. (Epidemics) A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and noninfected persons, who are assumed to move freely among each other. Set up the model. Find the equilibrium solutions and indicate their stability or instability. Solve the ODE. Find the limit of the proportion of infected persons as $t \to \infty$ and explain what it means.

31. (Extinction vs. unlimited growth) If in a population $y(t)$ the death rate is proportional to the population, and the birth rate is proportional to the chance encounters of meeting mates for reproduction, what will the model be? Without solving, find out what will eventually happen to a small initial population. To a large one. Then solve the model.

32. (Harvesting renewable resources. Fishing) Suppose that the population $y(t)$ of a certain kind of fish is given by the logistic equation (8), and fish are caught at a rate Hy proportional to y. Solve this so-called *Schaefer model.* Find the equilibrium solutions y_1 and y_2 (> 0) when $H < A$. The expression $Y = Hy_2$ is called the **equilibrium harvest** or **sustainable yield** *corresponding to H.* Why?

33. (Harvesting) In Prob. 32 find and graph the solution satisfying $y(0) = 2$ when (for simplicity) $A = B = 1$ and $H = 0.2$. What is the limit? What does it mean? What if there were no fishing?

34. (Intermittent harvesting) In Prob. 32 assume that you fish for 3 years, then fishing is banned for the next 3 years. Thereafter you start again. And so on. This is called *intermittent harvesting.* Describe qualitatively how the population will develop if intermitting is continued periodically. Find and graph the solution for the first 9 years, assuming that $A = B = 1$, $H = 0.2$, and $y(0) = 2$.

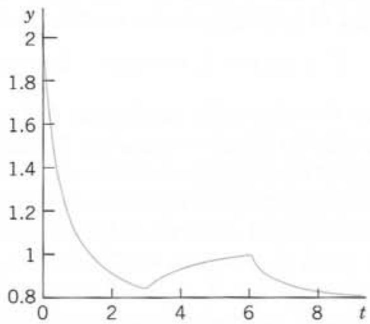


Fig. 21. Fish population in Problem 34

35. (Harvesting) If a population of mice (in multiples of 1000) follows the logistic law with $A = 1$ and $B = 0.25$, and if owls catch at a time rate of 10% of the population present, what is the model, its equilibrium harvest for that catch, and its solution?

36. (Harvesting) Do you save work in Prob. 34 if you first transform the ODE to a linear ODE? Do this transformation. Solve the resulting ODE. Does the resulting $y(t)$ agree with that in Prob. 34?

37–40 GENERAL PROPERTIES OF LINEAR ODEs

These properties are of practical and theoretical importance because they enable us to obtain new solutions from given ones. Thus in modeling, whenever possible, we prefer linear ODEs over nonlinear ones, which have no similar properties.

Show that nonhomogeneous linear ODEs (1) and homogeneous linear ODEs (2) have the following properties. Illustrate each property by a calculation for two or three equations of your choice. Give proofs.

37. The sum $y_1 + y_2$ of two solutions y_1 and y_2 of the homogeneous equation (2) is a solution of (2), and so is a scalar multiple ay_1 for any constant a. These properties are not true for (1)!

38. $y = 0$ (that is, $y(x) = 0$ for all x, also written $y(x) \equiv 0$) is a solution of (2) [not of (1) if $r(x) \neq 0$!], called the **trivial solution.**

39. The sum of a solution of (1) and a solution of (2) is a solution of (1).

40. The difference of two solutions of (1) is a solution of (2).

41. If y_1 is a solution of (1), what can you say about cy_1?

42. If y_1 and y_2 are solutions of $y_1' + py_1 = r_1$ and $y_2' + py_2 = r_2$, respectively (with the same p!), what can you say about the sum $y_1 + y_2$?

43. CAS EXPERIMENT. (a) Solve the ODE $y' - y/x = -x^{-1}\cos(1/x)$. Find an initial condition for which the arbitrary constant is zero. Graph the resulting particular solution, experimenting to obtain a good figure near $x = 0$.

(b) Generalizing (a) from $n = 1$ to arbitrary n, solve the ODE $y' - ny/x = -x^{n-2}\cos(1/x)$. Find an initial condition as in (a), and experiment with the graph.

44. TEAM PROJECT. Riccati Equation, Clairaut Equation. A **Riccati equation** is of the form

$$(11) \qquad y' + p(x)y = g(x)y^2 + h(x).$$

A **Clairaut equation** is of the form

$$(12) \qquad y = xy' + g(y').$$

(a) Apply the transformation $y = Y + 1/u$ to the Riccati equation (11), where Y is a solution of (11), and obtain for u the linear ODE $u' + (2Yg - p)u = -g$. Explain the effect of the transformation by writing it as $y = Y + v$, $v = 1/u$.

(b) Show that $y = Y = x$ is a solution of $y' - (2x^3 + 1)y = -x^2y^2 - x^4 - x + 1$ and solve this Riccati equation, showing the details.

(c) Solve $y' + (3 - 2x^2 \sin x)y = -y^2 \sin x + 2x + 3x^2 - x^4 \sin x$, using (and verifying) that $y = x^2$ is a solution.

(d) By working "backward" from the u-equation find further Riccati equations that have relatively simple solutions.

(e) Solve the Clairaut equation $y = xy' + 1/y'$. *Hint.* Differentiate this ODE with respect to x.

(f) Solve the Clairaut equation $y'^2 - xy' + y = 0$ in Prob. 16 of Problem Set 1.1.

(g) Show that the Clairaut equation (12) has as solutions a family of straight lines $y = cx + g(c)$ and a singular solution determined by $g'(s) = -x$, where $s = y'$, that forms the envelope of that family.

45. (Variation of parameter) Another method of obtaining (4) results from the following idea. Write (3) as cy^*, where y^* is the exponential function, which is a solution of the homogeneous linear ODE $y^{*\prime} + py^* = 0$. Replace the arbitrary constant c in (3) with a function u to be determined so that the resulting function $y = uy^*$ is a solution of the nonhomogeneous linear ODE $y' + py = r$.

46. TEAM PROJECT. Transformations of ODEs. We have transformed ODEs to separable form, to exact form, and to linear form. The purpose of such transformations is an extension of solution methods to larger classes of ODEs. Describe the key idea of each of these transformations and give three typical examples of your choice for each transformation, showing each step (not just the transformed ODE).

1.6 Orthogonal Trajectories. *Optional*

An important type of problem in physics or geometry is to find a family of curves that intersect a given family of curves at right angles. The new curves are called **orthogonal trajectories** of the given curves (and conversely). Examples are curves of equal temperature (*isotherms*) and curves of heat flow, curves of equal altitude (*contour lines*) on a map and curves of steepest descent on that map, curves of equal potential (*equipotential curves,* curves of equal voltage—the concentric circles in Fig. 22), and curves of electric force (the straight radial segments in Fig. 22).

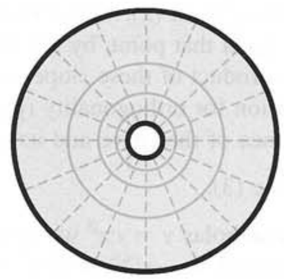


Fig. 22. Equipotential lines and curves of electric force (dashed) between two concentric (black) circles (cylinders in space)

Here the **angle of intersection** between two curves is defined to be the angle between the tangents of the curves at the intersection point. *Orthogonal* is another word for *perpendicular.*

In many cases orthogonal trajectories can be found by using ODEs, as follows. Let

$$G(x, y, c) = 0 \tag{1}$$

be a given family of curves in the xy-plane, where each curve is specified by some value of c. This is called a **one-parameter family of curves,** and c is called the *parameter* of the family. For instance, a one-parameter family of quadratic parabolas is given by (Fig. 23)

$$y = cx^2 \qquad \text{or, written as in (1),} \qquad G(x, y, c) = y - cx^2 = 0.$$

Step 1. Find an ODE for which the given family is a general solution. Of course, this ODE must no longer contain the parameter c. In our example we solve algebraically for c and then differentiate and simplify; thus,

$$\frac{y}{x^2} = c, \qquad \frac{y'x^2 - 2xy}{x^4} = 0,$$

hence

$$y' = \frac{2y}{x}.$$

The last of these equations is the ODE of the given family of curves. It is of the form

$$y' = f(x, y). \tag{2}$$

Step 2. Write down the ODE of the orthogonal trajectories, that is, the ODE whose general solution gives the orthogonal trajectories of the given curves. This ODE is

$$\tilde{y}' = -\frac{1}{f(x, \tilde{y})} \tag{3}$$

with the same f as in (2). Why? Well, a given curve passing through a point (x_0, y_0) has slope $f(x_0, y_0)$ at that point, by (2). The trajectory through (x_0, y_0) has slope $-1/f(x_0, y_0)$ by (3). The product of these slopes is -1, as we see. From calculus it is known that this is the condition for orthogonality (perpendicularity) of two straight lines (the tangents at (x_0, y_0)), hence of the curve and its orthogonal trajectory at (x_0, y_0).

Step 3. Solve (3).

For our parabolas $y = cx^2$ we have $y' = 2y/x$. Hence their orthogonal trajectories are obtained from $\tilde{y}' = -x/2\tilde{y}$ or $2\tilde{y}\tilde{y}' + x = 0$. By integration, $\tilde{y}^2 + \frac{1}{2}x^2 = c^*$. These are the ellipses in Fig. 23 with semi-axes $\sqrt{2c^*}$ and $\sqrt{c^*}$. Here, $c^* > 0$ because $c^* = 0$ gives just the origin, and $c^* < 0$ gives no real solution at all.

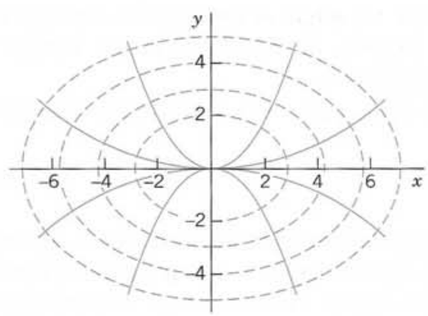


Fig. 23. Parabolas and orthogonal trajectories (ellipses) in the text

PROBLEM SET 1.6

1–12 ORTHOGONAL TRAJECTORIES

Sketch or graph some of the given curves. Guess what their orthogonal trajectories may look like. Find these trajectories.

(Show the details of your work.)

1. $y = 4x + c$

2. $y = c/x$

3. $y = cx$

4. $y^2 = 2x^2 + c$

5. $x^2y = c$

6. $y = ce^{-3x}$

7. $y = ce^{x^2/2}$

8. $x^2 - y^2 = c$

9. $4x^2 + y^2 = c$

10. $x = c\sqrt{y}$

11. $x = ce^{y/4}$

12. $x^2 + (y - c)^2 = c^2$

13–15 OTHER FORMS OF THE ODEs (2) AND (3)

13. (y as independent variable) Show that (3) may be written $dx/d\tilde{y} = -f(x, \tilde{y})$. Use this form to find the orthogonal trajectories of $y = 2x + ce^{-x}$.

14. (Family $g(x, y) = c$) Show that if a family is given as $g(x, y) = c$, then the orthogonal trajectories can be obtained from the following ODE, and use the latter to solve Prob. 6 written in the form $g(x, y) = c$.

$$\frac{d\tilde{y}}{dx} = \frac{\partial g/\partial \tilde{y}}{\partial g/\partial x}$$

15. (Cauchy–Riemann equations) Show that for a family $u(x, y) = c = const$ the orthogonal trajectories $v(x, y) = c^* = const$ can be obtained from the following *Cauchy–Riemann equations* (which are basic in complex analysis in Chap. 13) and use them to find the orthogonal trajectories of $e^x \sin y = const$. (Here, subscripts denote partial derivatives.)

$$u_x = v_y, \qquad u_y = -v_x$$

16–20 APPLICATIONS

16. (Fluid flow) Suppose that the *streamlines* of the flow (paths of the particles of the fluid) in Fig. 24 are $\Psi(x, y) = xy = const$. Find their orthogonal trajectories (called **equipotential lines,** for reasons given in Sec. 18.4).

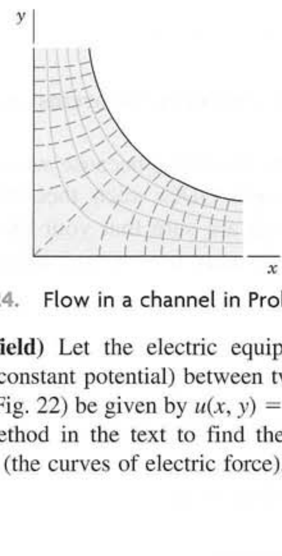


Fig. 24. Flow in a channel in Problem 16

17. (Electric field) Let the electric equipotential lines (curves of constant potential) between two concentric cylinders (Fig. 22) be given by $u(x, y) = x^2 + y^2 = c$. Use the method in the text to find their orthogonal trajectories (the curves of electric force).

18. (Electric field) The lines of electric force of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are the circles through $(-1, 0)$ and $(1, 0)$. Show that these circles are given by $x^2 + (y - c)^2 = 1 + c^2$. Show that the **equipotential lines** (orthogonal trajectories of those circles) are the circles given by $(x + c^*)^2 + \tilde{y}^2 = c^{*2} - 1$ (dashed in Fig. 25).

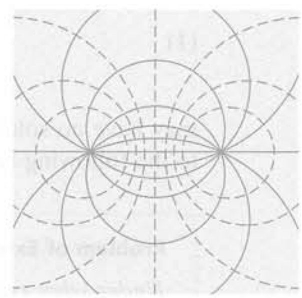


Fig. 25. Electric field in Problem 18

19. (Temperature field) Let the **isotherms** (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by $4x^2 + 9y^2 = c$. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

20. TEAM PROJECT. Conic Sections. (A) State the main steps of the present method of obtaining orthogonal trajectories.

(B) Find conditions under which the orthogonal trajectories of families of ellipses $x^2/a^2 + y^2/b^2 = c$ are again conic sections. Illustrate your result graphically by sketches or by using your CAS. What happens if $a \to 0$? If $b \to 0$?

(C) Investigate families of hyperbolas $x^2/a^2 - y^2/b^2 = c$ in a similar fashion.

(D) Can you find more complicated curves for which you get ODEs that you can solve? Give it a try.

1.7 Existence and Uniqueness of Solutions

The initial value problem

$$|y'| + |y| = 0, \qquad y(0) = 1$$

has no solution because $y = 0$ (that is, $y(x) = 0$ for all x) is the only solution of the ODE. The initial value problem

$$y' = 2x, \qquad y(0) = 1$$

has precisely one solution, namely, $y = x^2 + 1$. The initial value problem

$$xy' = y - 1, \qquad y(0) = 1$$

has infinitely many solutions, namely, $y = 1 + cx$, where c is an arbitrary constant because $y(0) = 1$ for all c.

From these examples we see that an **initial value problem**

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

may have no solution, precisely one solution, or more than one solution. This fact leads to the following two fundamental questions.

Problem of Existence

Under what conditions does an initial value problem of the form (1) *have at least one solution* (*hence one or several solutions*)?

Problem of Uniqueness

Under what conditions does that problem have at most one solution (*hence excluding the case that is has more than one solution*)?

Theorems that state such conditions are called **existence theorems** and **uniqueness theorems,** respectively.

Of course, for our simple examples we need no theorems because we can solve these examples by inspection; however, for complicated ODEs such theorems may be of considerable practical importance. Even when you are sure that your physical or other system behaves uniquely, occasionally your model may be oversimplified and may not give a faithful picture of the reality.

THEOREM 1

Existence Theorem

Let the right side $f(x, y)$ of the ODE in the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

be continuous at all points (x, y) in some rectangle

$$R\colon |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in R; that is, there is a number K such that*

$$|f(x, y)| \leqq K \qquad \textit{for all } (x, y) \textit{ in } R. \tag{2}$$

Then the initial value problem (1) *has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$; here, α is the smaller of the two numbers a and b/K.*

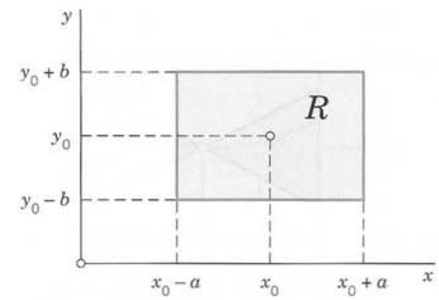


Fig. 26. Rectangle R in the existence and uniqueness theorems

(*Example of Boundedness.* The function $f(x, y) = x^2 + y^2$ is bounded (with $K = 2$) in the square $|x| < 1$, $|y| < 1$. The function $f(x, y) = \tan(x + y)$ is not bounded for $|x + y| < \pi/2$. Explain!)

THEOREM 2

Uniqueness Theorem

Let f and its partial derivative $f_y = \partial f/\partial y$ be continuous for all (x, y) in the rectangle R (Fig. 26) *and bounded, say,*

$$\text{(3)} \qquad \text{(a)}\quad |f(x, y)| \leqq K, \qquad \text{(b)}\quad |f_y(x, y)| \leqq M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at most one solution $y(x)$. Thus, by Theorem* 1, *the problem has precisely one solution. This solution exists at least for all x in that subinterval $|x - x_0| < \alpha$.*

Understanding These Theorems

These two theorems take care of almost all practical cases. Theorem 1 says that if $f(x, y)$ is continuous in some region in the xy-plane containing the point (x_0, y_0), then the initial value problem (1) has at least one solution.

Theorem 2 says that if, moreover, the partial derivative $\partial f/\partial y$ of f with respect to y exists and is continuous in that region, then (1) can have at most one solution; hence, by Theorem 1, it has precisely one solution.

Read again what you have just read—these are entirely new ideas in our discussion.

Proofs of these theorems are beyond the level of this book (see Ref. [A11] in App. 1); however, the following remarks and examples may help you to a good understanding of the theorems.

Since $y' = f(x, y)$, the condition (2) implies that $|y'| \leqq K$; that is, the slope of any solution curve $y(x)$ in R is at least $-K$ and at most K. Hence a solution curve that passes through the point (x_0, y_0) must lie in the colored region in Fig. 27 on the next page bounded by the lines l_1 and l_2 whose slopes are $-K$ and K, respectively. Depending on the form of R, two different cases may arise. In the first case, shown in Fig. 27a, we have $b/K \geqq a$ and therefore $\alpha = a$ in the existence theorem, which then asserts that the solution exists for all x between $x_0 - a$ and $x_0 + a$. In the second case, shown in Fig. 27b, we have $b/K < a$. Therefore, $\alpha = b/K < a$, and all we can conclude from the theorems is that the solution exists for all x between $x_0 - b/K$ and $x_0 + b/K$. For larger or smaller x's the solution curve may leave the rectangle R, and since nothing is assumed about f outside R, nothing can be concluded about the solution for those larger or smaller x's; that is, for such x's the solution may or may not exist—we don't know.

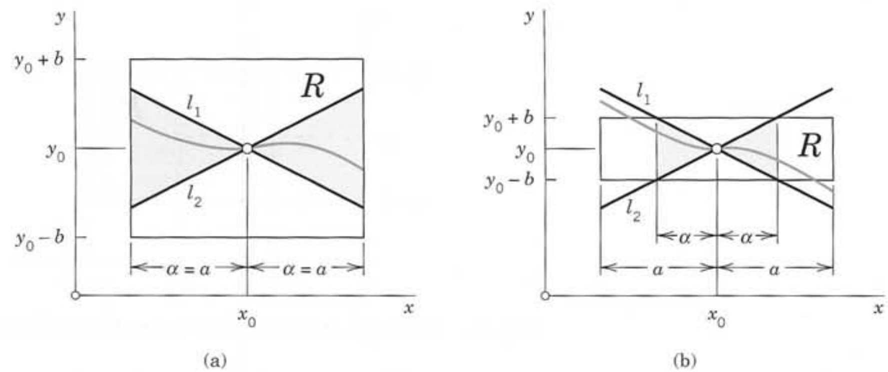


Fig. 27. The condition (2) of the existence theorem. (a) First case. (b) Second case

Let us illustrate our discussion with a simple example. We shall see that our choice of a rectangle R with a large base (a long x-interval) will lead to the case in Fig. 27b.

EXAMPLE 1 Choice of a Rectangle

Consider the initial value problem

$$y' = 1 + y^2, \qquad y(0) = 0$$

and take the rectangle R; $|x| < 5$, $|y| < 3$. Then $a = 5$, $b = 3$, and

$$|f(x, y)| = |1 + y^2| \leqq K = 10,$$

$$\left| \frac{\partial f}{\partial y} \right| = 2|y| \leqq M = 6,$$

$$\alpha = \frac{b}{K} = 0.3 < a.$$

Indeed, the solution of the problem is $y = \tan x$ (see Sec. 1.3, Example 1). This solution is discontinuous at $\pm\pi/2$, and there is no *continuous* solution valid in the entire interval $|x| < 5$ from which we started. ■

The conditions in the two theorems are sufficient conditions rather than necessary ones, and can be lessened. In particular, by the mean value theorem of differential calculus we have

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1) \left. \frac{\partial f}{\partial y} \right|_{y=\tilde{y}}$$

where (x, y_1) and (x, y_2) are assumed to be in R, and $\tilde{y}$ is a suitable value between y_1 and y_2. From this and (3b) it follows that

$$|f(x, y_2) - f(x, y_1)| \leqq M|y_2 - y_1|. \tag{4}$$

It can be shown that (3b) may be replaced by the weaker condition (4), which is known as a **Lipschitz condition.**[7] However, continuity of $f(x, y)$ is not enough to guarantee the *uniqueness* of the solution. This may be illustrated by the following example.

EXAMPLE 2 Nonuniqueness

The initial value problem

$$y' = \sqrt{|y|}, \qquad y(0) = 0$$

has the two solutions

$$y \equiv 0 \qquad \text{and} \qquad y^* = \begin{cases} x^2/4 & \text{if } x \geqq 0 \\ -x^2/4 & \text{if } x < 0 \end{cases}$$

although $f(x, y) = \sqrt{|y|}$ is continuous for all y. The Lipschitz condition (4) is violated in any region that includes the line $y = 0$, because for $y_1 = 0$ and positive y_2 we have

$$\text{(5)} \qquad \frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, \qquad (\sqrt{y_2} > 0)$$

and this can be made as large as we please by choosing y_2 sufficiently small, whereas (4) requires that the quotient on the left side of (5) should not exceed a fixed constant M. ■

PROBLEM SET 1.7

1. **(Vertical strip)** If the assumptions of Theorems 1 and 2 are satisfied not merely in a rectangle but in a vertical infinite strip $|x - x_0| < a$, in what interval will the solution of (1) exist?
2. **(Existence?)** Does the initial value problem $(x - 1)y' = 2y$, $y(1) = 1$ have a solution? Does your result contradict our present theorems?
3. **(Common points)** Can two solution curves of the same ODE have a common point in a rectangle in which the assumptions of the present theorems are satisfied?
4. **(Change of initial condition)** What happens in Prob. 2 if you replace $y(1) = 1$ with $y(1) = k$?
5. **(Linear ODE)** If p and r in $y' + p(x)y = r(x)$ are continuous for all x in an interval $|x - x_0| \leqslant a$, show that $f(x, y)$ in this ODE satisfies the conditions of our present theorems, so that a corresponding initial value problem has a unique solution. Do you actually need these theorems for this ODE?
6. **(Three possible cases)** Find all initial conditions such that $(x^2 - 4x)y' = (2x - 4)y$ has no solution, precisely one solution, and more than one solution.
7. **(Length of x-interval)** In most cases the solution of an initial value problem (1) exists in an x-interval larger than that guaranteed by the present theorems. Show this fact for $y' = 2y^2$, $y(1) = 1$ by finding the best possible α (choosing b optimally) and comparing the result with the actual solution.
8. **PROJECT. Lipschitz Condition. (A)** State the definition of a Lipschitz condition. Explain its relation to the existence of a partial derivative. Explain its significance in our present context. Illustrate your statements by examples of your own.

   (B) Show that for a ***linear*** ODE $y' + p(x)y = r(x)$ with continuous p and r in $|x - x_0| \leqq a$ a Lipschitz condition holds. This is remarkable because it means that for a *linear* ODE the continuity of $f(x, y)$ guarantees not only the existence but also the uniqueness of the solution of an initial value problem. (Of course, this also follows directly from (4) in Sec. 1.5.)

   (C) Discuss the uniqueness of solution for a few simple ODEs that you can solve by one of the methods considered, and find whether a Lipschitz condition is satisfied.
9. **(Maximum α)** What is the largest possible α in Example 1 in the text?
10. **CAS PROJECT. Picard Iteration. (A)** Show that by integrating the ODE in (1) and observing the initial condition you obtain

$$\text{(6)} \qquad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

[7]RUDOLF LIPSCHITZ (1832–1903), German mathematician. Lipschitz and similar conditions are important in modern theories, for instance, in partial differential equations.

This form (6) of (1) suggests **Picard's iteration method**[8], which is defined by

$$(7) \quad y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \quad n = 1, 2, \cdots.$$

It gives approximations $y_1, y_2, y_3, \cdots$ of the unknown solution y of (1). Indeed, you obtain y_1 by substituting $y = y_0$ on the right and integrating—this is the first step—, then y_2 by substituting $y = y_1$ on the right and integrating—this is the second step—, and so on. Write a program of the iteration that gives a printout of the first approximations $y_0, y_1, \cdots, y_N$ as well as their graphs on common axes. Try your program on two initial value problems of your own choice.

(B) Apply the iteration to $y' = x + y$, $y(0) = 0$. Also solve the problem exactly.

(C) Apply the iteration to $y' = 2y^2$, $y(0) = 1$. Also solve the problem exactly.

(D) Find all solutions of $y' = 2\sqrt{y}$, $y(1) = 0$. Which of them does Picard's iteration approximate?

(E) Experiment with the conjecture that Picard's iteration converges to the solution of the problem for any initial choice of y in the integrand in (7) (leaving y_0 outside the integral as it is). Begin with a simple ODE and see what happens. When you are reasonably sure, take a slightly more complicated ODE and give it a try.

CHAPTER 1 REVIEW QUESTIONS AND PROBLEMS

1. Explain the terms *ordinary differential equation (ODE), partial differential equation (PDE), order, general solution,* and *particular solution.* Give examples. Why are these concepts of importance?
2. What is an initial condition? How is this condition used in an initial value problem?
3. What is a homogeneous linear ODE? A nonhomogeneous linear ODE? Why are these equations simpler than nonlinear ODEs?
4. What do you know about direction fields and their practical importance?
5. Give examples of mechanical problems that lead to ODEs.
6. Why do electric circuits lead to ODEs?
7. Make a list of the solution methods considered. Explain each method with a few short sentences and illustrate it by a typical example.
8. Can certain ODEs be solved by more than one method? Give three examples.
9. What are integrating factors? Explain the idea. Give examples.
10. Does every first-order ODE have a solution? A general solution? What do you know about uniqueness of solutions?

11–14 DIRECTION FIELDS

Graph a direction field (by a CAS or by hand) and sketch some of the solution curves. Solve the ODE exactly and compare.

11. $y' = 1 + 4y^2$
12. $y' = 3y - 2x$
13. $y' = 4y - y^2$
14. $y' = 16x/y$

15–26 GENERAL SOLUTION

Find the general solution. Indicate which method in this chapter you are using. Show the details of your work.

15. $y' = x^2(1 + y^2)$
16. $y' = x(y - x^2 + 1)$
17. $yy' + xy^2 = x$
18. $-\pi \sin \pi x \cosh 3y\, dx + 3 \cos \pi x \sinh 3y\, dy = 0$
19. $y' + y \sin x = \sin x$
20. $y' - y = 1/y$
21. $3 \sin 2y\, dx + 2x \cos 2y\, dy = 0$
22. $xy' = x \tan (y/x) + y$
23. $(y \cos xy - 2x)\, dx + (x \cos xy + 2y)\, dy = 0$
24. $xy' = (y - 2x)^2 + y$ (Set $y - 2x = z$.)
25. $\sin (y - x)\, dx + [\cos (y - x) - \sin (y - x)]\, dy = 0$
26. $xy' = (y/x)^3 + y$

27–32 INITIAL VALUE PROBLEMS

Solve the following initial value problems. Indicate the method used. Show the details of your work.

27. $yy' + x = 0, \quad y(3) = 4$
28. $y' - 3y = -12y^2, \quad y(0) = 2$
29. $y' = 1 + y^2, \quad y(\frac{1}{4}\pi) = 0$
30. $y' + \pi y = 2b \cos \pi x, \quad y(0) = 0$
31. $(2xy^2 - \sin x)\, dx + (2 + 2x^2 y)\, dy = 0, \; y(0) = 1$
32. $[2y + y^2/x + e^x(1 + 1/x)]\, dx + (x + 2y)\, dy = 0, \; y(1) = 1$

[8]EMILE PICARD (1856–1941), French mathematician, also known for his important contributions to complex analysis (see Sec. 16.2 for his famous theorem). Picard used his method to prove Theorems 1 and 2 as well as the convergence of the sequence (7) to the solution of (1). In precomputer times the iteration was of little *practical* value because of the integrations.

33–43 APPLICATIONS, MODELING

33. (Heat flow) If the isotherms in a region are $x^2 - y^2 = c$, what are the curves of heat flow (assuming orthogonality)?

34. (Law of cooling) A thermometer showing 10°C is brought into a room whose temperature is 25°C. After 5 minutes it shows 20°C. When will the thermometer practically reach the room temperature, say, 24.9°C?

35. (Half-life) If 10% of a radioactive substance disintegrates in 4 days, what is its half-life?

36. (Half-life) What is the half-life of a substance if after 5 days, 0.020 g is present and after 10 days, 0.015 g?

37. (Half-life) When will 99% of the substance in Prob. 35 have disintegrated?

38. (Air circulation) In a room containing 20 000 ft^3 of air, 600 ft^3 of fresh air flows in per minute, and the mixture (made practically uniform by circulating fans) is exhausted at a rate of 600 cubic feet per minute (cfm). What is the amount of fresh air $y(t)$ at any time if $y(0) = 0$? After what time will 90% of the air be fresh?

39. (Electric field) If the equipotential lines in a region of the xy-plane are $4x^2 + y^2 = c$, what are the curves of the electrical force? Sketch both families of curves.

40. (Chemistry) In a bimolecular reaction $A + B \to M$, a moles per liter of a substance A and b moles per liter of a substance B are combined. Under constant temperature the rate of reaction is

$$y' = k(a - y)(b - y) \qquad \textbf{(Law of mass action);}$$

that is, y' is proportional to the product of the concentrations of the substances that are reacting, where $y(t)$ is the number of moles per liter which have reacted after time t. Solve this ODE, assuming that $a \neq b$.

41. (Population) Find the population $y(t)$ if the birth rate is proportional to $y(t)$ and the death rate is proportional to the square of $y(t)$.

42. (Curves) Find all curves in the first quadrant of the xy-plane such that for every tangent, the segment between the coordinate axes is bisected by the point of tangency. (Make a sketch.)

43. (Optics) Lambert's law of absorption[9] states that the absorption of light in a thin transparent layer is proportional to the thickness of the layer and to the amount of light incident on that layer. Formulate this law as an ODE and solve it.

SUMMARY OF CHAPTER 1
First-Order ODEs

This chapter concerns **ordinary differential equations (ODEs) of first order** and their applications. These are equations of the form

$$(1) \qquad F(x, y, y') = 0 \qquad \text{or in explicit form} \qquad y' = f(x, y)$$

involving the derivative $y' = dy/dx$ of an unknown function y, given functions of x, and, perhaps, y itself. If the independent variable x is time, we denote it by t.

In Sec. 1.1 we explained the basic concepts and the process of **modeling,** that is, of expressing a physical or other problem in some mathematical form and solving it. Then we discussed the method of direction fields (Sec. 1.2), solution methods and models (Secs. 1.3–1.6), and, finally, ideas on existence and uniqueness of solutions (Sec. 1.7).

[9]JOHANN HEINRICH LAMBERT (1728–1777), German physicist and mathematician.

A first-order ODE usually has a **general solution,** that is, a solution involving an arbitrary constant, which we denote by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem**

$$y' = f(x, y), \qquad y(x_0,) = y_0 \quad (x_0, y_0 \text{ given numbers}) \tag{2}$$

and its solution is a **particular solution** of the ODE. Geometrically, a general solution represents a family of curves, which can be graphed by using **direction fields** (Sec. 1.2). And each particular solution corresponds to one of these curves.

A **separable ODE** is one that we can put into the form

$$g(y)\, dy = f(x)\, dx \qquad \text{(Sec. 1.3)} \tag{3}$$

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides.

An **exact ODE** is of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \qquad \text{(Sec. 1.4)} \tag{4}$$

where $M\, dx + N\, dy$ is the **differential**

$$du = u_x\, dx + u_y\, dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. This method extends to nonexact ODEs that can be made exact by multiplying them by some function $F(x, y)$, called an **integrating factor** (Sec. 1.4).

Linear ODEs

$$y' + p(x)y = r(x) \tag{5}$$

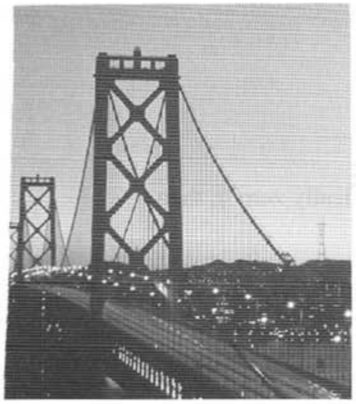
are very important. Their solutions are given by the integral formula (4), Sec. 1.5. Certain nonlinear ODEs can be transformed to linear form in terms of new variables. This holds for the **Bernoulli equation**

$$y' + p(x)y = g(x)y^a \qquad \text{(Sec. 1.5).}$$

Applications and **modeling** are discussed throughout the chapter, in particular in Secs. 1.1, 1.3, 1.5 (**population dynamics,** etc.), and 1.6 (**trajectories**).

Picard's **existence** and **uniqueness theorems** are explained in Sec. 1.7 (and *Picard's iteration* in Problem Set 1.7).

Numeric methods for first-order ODEs can be studied in Secs. 21.1 and 21.2 immediately after this chapter, as indicated in the chapter opening.



CHAPTER 2

Second-Order Linear ODEs

Ordinary differential equations (ODEs) may be divided into two large classes, **linear ODEs** and **nonlinear ODEs.** Whereas nonlinear ODEs of second (and higher) order generally are difficult to solve, linear ODEs are much simpler because various properties of their solutions can be characterized in a general way, and there are standard methods for solving many of these equations.

Linear ODEs ***of the second order*** are the most important ones because of their applications in mechanical and electrical engineering (Secs. 2.4, 2.8, 2.9). And their theory is typical of that of all linear ODEs, but the formulas are simpler than for higher order equations. Also the transition to higher order (in Chap. 3) will be almost immediate.

This chapter includes the derivation of general and particular solutions, the latter in connection with initial value problems.

(Boundary value problems follow in Chap. 5, which also contains solution methods for Legendre's, Bessel's, and the hypergeometric equations.)

COMMENT. ***Numerics for second-order ODEs can be studied immediately after this chapter.*** See Sec. 21.3, which is independent of other sections in Chaps. 19–21.

Prerequisite: Chap. 1, in particular, Sec. 1.5.
Sections that may be omitted in a shorter course: 2.3, 2.9, 2.10.
References and Answers to Problems: App. 1 Part A, and App. 2.

2.1 Homogeneous Linear ODEs of Second Order

We have already considered first-order linear ODEs (Sec. 1.5) and shall now define and discuss linear ODEs of second order. These equations have important engineering applications, especially in connection with mechanical and electrical vibrations (Secs. 2.4, 2.8, 2.9) as well as in wave motion, heat conduction, and other parts of physics, as we shall see in Chap. 12.

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

and **nonlinear** if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in y and its derivatives,* whereas the functions p, q, and r on the right may be any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term, which is practical.

If $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and is called **homogeneous.** If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous.** This is similar to Sec. 1.5.

For instance, a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0, \qquad \text{in standard form} \qquad y'' + \frac{1}{x} y' + y = 0.$$

An example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs.

Solutions are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''. Examples are given below.

Homogeneous Linear ODEs: Superposition Principle

Sections 2.1–2.6 will be devoted to **homogeneous** linear ODEs (2) and the remaining sections of the chapter to nonhomogeneous linear ODEs.

Linear ODEs have a rich solution structure. For the homogeneous equation the backbone of this structure is the *superposition principle* or *linearity principle,* which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants. Of course, this is a great advantage of homogeneous linear ODEs. Let us first discuss an example.

EXAMPLE 1 Homogeneous Linear ODEs: Superposition of Solutions

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all x. We verify this by differentiation and substitution. We obtain $(\cos x)'' = -\cos x$; hence

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

Similarly for $y = \sin x$ (verify!). We can go an important step further. We multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by, say, -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x) = -4.7 \cos x + 2 \sin x + 4.7 \cos x - 2 \sin x = 0. \quad \blacksquare$$

In this example we have obtained from y_1 (= cos x) and y_2 (= sin x) a function of the form

$$y = c_1y_1 + c_2y_2 \qquad (c_1, c_2 \text{ arbitrary constants}). \tag{3}$$

This is called a **linear combination** of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the **superposition principle** or **linearity principle.**

THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

PROOF Let y_1 and y_2 be solutions of (2) on I. Then by substituting $y = c_1y_1 + c_2y_2$ and its derivatives into (2), and using the familiar rule $(c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2'$, etc., we get

$$\begin{aligned} y'' + py' + qy &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= c_1y_1'' + c_2y_2'' + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0, \end{aligned}$$

since in the last line, $(\cdots) = 0$ because y_1 and y_2 are solutions, by assumption. This shows that y is a solution of (2) on I. ■

CAUTION! Don't forget that this highly important theorem holds for *homogeneous linear* ODEs only but ***does not hold*** for nonhomogeneous linear or nonlinear ODEs, as the following two examples illustrate.

EXAMPLE 2 A Nonhomogeneous Linear ODE

Verify by substitution that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear ODE

$$y'' + y = 1,$$

but their sum is not a solution. Neither is, for instance, $2(1 + \cos x)$ or $5(1 + \sin x)$. ■

EXAMPLE 3 A Nonlinear ODE

Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy' = 0,$$

but their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by -1! ■

Initial Value Problem. Basis. General Solution

Recall from Chap. 1 that for a first-order ODE, an *initial value problem* consists of the ODE and one *initial condition* $y(x_0) = y_0$. The initial condition is used to determine the *arbitrary constant c* in the *general solution* of the ODE. This results in a unique solution, as we need it in most applications. That solution is called a *particular solution* of the ODE. These ideas extend to second-order equations as follows.

For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

$$y(x_0) = K_0, \qquad y'(x_0) = K_1. \tag{4}$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

$$y = c_1 y_1 + c_2 y_2 \tag{5}$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE, with "suitable" to be explained after the next example. This results in a unique solution, passing through the point (x_0, K_0) with K_1 as the tangent direction (the slope) at that point. That solution is called a **particular solution** of the ODE (2).

EXAMPLE 4 Initial Value Problem

Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution. ***Step 1. General solution.*** The functions $\cos x$ and $\sin x$ are solutions of the ODE (by Example 1), and we take

$$y = c_1 \cos x + c_2 \sin x.$$

This will turn out to be a general solution as defined below.

Step 2. Particular solution. We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, since $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3.0 \qquad \text{and} \qquad y'(0) = c_2 = -0.5.$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x.$$

Figure 28 shows that at $x = 0$ it has the value 3.0 and the slope -0.5, so that its tangent intersects the x-axis at $x = 3.0/0.5 = 6.0$. (The scales on the axes differ!) ■

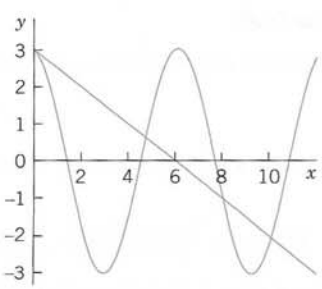


Fig. 28. Particular solution and initial tangent in Example 4

Observation. Our choice of y_1 and y_2 was general enough to satisfy both initial conditions. Now let us take instead two proportional solutions $y_1 = \cos x$ and

$y_2 = k \cos x$, so that $y_1/y_2 = 1/k = const.$ Then we can write $y = c_1y_1 + c_2y_2$ in the form

$$y = c_1 \cos x + c_2(k \cos x) = C \cos x \qquad \text{where} \qquad C = c_1 + c_2k.$$

Hence we are no longer able to satisfy two initial conditions with only one arbitrary constant C. Consequently, in defining the concept of a general solution, we must exclude proportionality. And we see at the same time why the concept of a general solution is of importance in connection with initial value problems.

DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of an ODE (2) on an open interval I is a solution (5) in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called a **basis** (or a **fundamental system**) of solutions of (2) on I.

A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (5).

For the definition of an *interval* see Sec. 1.1. Also, c_1 and c_2 must sometimes be restricted to some interval in order to avoid complex expressions in the solution. Furthermore, as usual, y_1 and y_2 are called *proportional* on I if for all x on I,

$$\text{(6)} \qquad \text{(a)} \quad y_1 = ky_2 \qquad \text{or} \qquad \text{(b)} \quad y_2 = ly_1$$

where k and l are numbers, zero or not. (Note that (a) implies (b) if and only if $k \neq 0$).

Actually, we can reformulate our definition of a basis by using a concept of general importance. Namely, two functions y_1 and y_2 are called **linearly independent** on an interval I where they are defined if

$$\text{(7)} \qquad k_1y_1(x) + k_2y_2(x) = 0 \qquad \text{everywhere on } I \text{ implies} \qquad k_1 = 0 \text{ and } k_2 = 0.$$

And y_1 and y_2 are called **linearly dependent** on I if (7) also holds for some constants k_1, k_2 not both zero. Then if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1} y_2 \qquad \text{or} \qquad y_2 = -\frac{k_1}{k_2} y_1.$$

In contrast, in the case of linear *independence* these functions are not proportional because then we cannot divide in (7). This gives the following

DEFINITION

Basis (Reformulated)

A **basis** of solutions of (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

If the coefficients p and q of (2) are continuous on some open interval I, then (2) has a general solution. It yields the unique solution of any initial value problem (2), (4). It includes all solutions of (2) on I; hence (2) has no *singular solutions* (solutions not obtainable from of a general solution; see also Problem Set 1.1). All this will be shown in Sec. 2.6.

EXAMPLE 5 Basis, General Solution, Particular Solution

$\cos x$ and $\sin x$ in Example 4 form a basis of solutions of the ODE $y'' + y = 0$ for all x because their quotient is $\cot x \neq const$ (or $\tan x \neq const$). Hence $y = c_1 \cos x + c_2 \sin x$ is a general solution. The solution $y = 3.0 \cos x - 0.5 \sin x$ of the initial value problem is a particular solution. ■

EXAMPLE 6 Basis, General Solution, Particular Solution

Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y'' - y = 0, \qquad y(0) = 6, \qquad y'(0) = -2.$$

Solution. $(e^x)'' - e^x = 0$ and $(e^{-x})'' - e^{-x} = 0$ shows that e^x and e^{-x} are solutions. They are not proportional, $e^x/e^{-x} = e^{2x} \neq const.$ Hence e^x, e^{-x} form a basis for all x. We now write down the corresponding general solution and its derivative and equate their values at 0 to the given initial conditions,

$$y = c_1e^x + c_2e^{-x}, \qquad y' = c_1e^x - c_2e^{-x}, \qquad y(0) = c_1 + c_2 = 6, \qquad y'(0) = c_1 - c_2 = -2.$$

By addition and subtraction, $c_1 = 2$, $c_2 = 4$, so that the *answer* is $y = 2e^x + 4e^{-x}$. This is the particular solution satisfying the two initial conditions. ■

Find a Basis if One Solution Is Known. Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of **reduction of order.**[1] We first show this method for an example and then in general.

EXAMPLE 7 Reduction of Order if a Solution Is Known. Basis

Find a basis of solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

Solution. Inspection shows that $y_1 = x$ is a solution because $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u, \qquad y'' = u''x + 2u'$$

into the ODE. This gives

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

[1]Credited to the great mathematician JOSEPH LOUIS LAGRANGE (1736–1813), who was born in Turin, of French extraction, got his first professorship when he was 19 (at the Military Academy of Turin), became director of the mathematical section of the Berlin Academy in 1766, and moved to Paris in 1787. His important major work was in the calculus of variations, celestial mechanics, general mechanics (*Mécanique analytique*, Paris, 1788), differential equations, approximation theory, algebra, and number theory.

This ODE is of first order in $v = u'$, namely, $(x^2 - x)v' + (x - 2)v = 0$. Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2 - x}\,dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx, \qquad \ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}.$$

We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ln|x| + 1.$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive x. ■

In this example we applied **reduction of order** to a homogeneous linear ODE [see (2)]

$$y'' + p(x)y' + q(x)y = 0.$$

Note that we now take the ODE in standard form, with y'', not $f(x)y''$—this is essential in applying our subsequent formulas. We assume a solution y_1 of (2) on an open interval I to be known and want to find a basis. For this we need a second linearly independent solution y_2 of (2) on I. To get y_2, we substitute

$$y = y_2 = uy_1, \qquad y' = y_2' = u'y_1 + uy_1', \qquad y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into (2). This gives

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0. \tag{8}$$

Collecting terms in u'', u', and u, we have

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0.$$

Now comes the main point. Since y_1 is a solution of (2), the expression in the last parentheses is zero. Hence u is gone, and we are left with an ODE in u' and u''. We divide this remaining ODE by y_1 and set $u' = U$, $u'' = U'$,

$$u'' + u'\,\frac{2y_1' + py_1}{y_1} = 0, \qquad \text{thus} \qquad U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0.$$

This is the desired first-order ODE, the reduced ODE. Separation of variables and integration gives

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx \qquad \text{and} \qquad \ln|U| = -2\ln|y_1| - \int p\,dx.$$

By taking exponents we finally obtain

$$U = \frac{1}{y_1^2}\,e^{-\int p\,dx}. \tag{9}$$

Here $U = u'$, so that $u = \int U\,dx$. Hence the desired second solution is

$$y_2 = y_1 u = y_1 \int U\,dx.$$

The quotient $y_2/y_1 = u = \int U\,dx$ cannot be constant (since $U > 0$), so that y_1 and y_2 form a basis of solutions.

PROBLEM SET 2.1

1–6 GENERAL SOLUTION. INITIAL VALUE PROBLEM

(More in the next problem set.) Verify by substitution that the given functions form a basis. Solve the given initial value problem. (Show the details of your work.)

1. $y'' - 16y = 0,\quad e^{4x}, e^{-4x},\quad y(0) = 3, y'(0) = 8$

2. $y'' + 25y = 0$, $\cos 5x$, $\sin 5x$, $y(0) = 0.8$, $y'(0) = -6.5$

3. $y'' + 2y' + 2y = 0,\quad e^{-x}\cos x, e^{-x}\sin x$, $y(0) = 1, y'(0) = -1$

4. $y'' - 6y' + 9y = 0$, e^{3x}, xe^{3x}, $y(0) = -1.4$, $y'(0) = 4.6$

5. $x^2y'' + xy' - 4y = 0$, x^2, x^{-2}, $y(1) = 11$, $y'(1) = -6$

6. $x^2y'' - 7xy' + 15y = 0$, x^3, x^5, $y(1) = 0.4$, $y'(1) = 1.0$

7–14 LINEAR INDEPENDENCE AND DEPENDENCE

Are the following functions linearly independent on the given interval?

7. x, $x \ln x$ $(0 < x < 10)$

8. $3x^2$, $2x^n$ $(0 < x < 1)$

9. e^{ax}, e^{-ax} (any interval)

10. $\cos^2 x$, $\sin^2 x$ (any interval)

11. $\ln x$, $\ln x^2$ $(x > 0)$

12. $x - 2$, $x + 2$ $(-2 < x < 2)$

13. $5 \sin x \cos x$, $3 \sin 2x$ $(x > 0)$

14. 0, $\sinh \pi x$ $(x > 0)$

REDUCTION OF ORDER is important because it gives a simpler ODE. A second-order ODE $F(x, y, y', y'') = 0$, linear or not, can be reduced to first order if y does not occur explicitly (Prob. 15) or if x does not occur explicitly (Prob. 16) or if the ODE is homogeneous linear and we know a solution (see the text).

15. (Reduction) Show that $F(x, y', y'') = 0$ can be reduced to first order in $z = y'$ (from which y follows by integration). Give two examples of your own.

16. (Reduction) Show that $F(y, y', y'') = 0$ can be reduced to a first-order ODE with y as the independent variable and $y'' = (dz/dy)z$, where $z = y'$; derive this by the chain rule. Give two examples.

17–22 Reduce to first order and solve (showing each step in detail).

17. $y'' = ky'$

18. $y'' = 1 + y'^2$

19. $yy'' = 4y'^2$

20. $xy'' + 2y' + xy = 0,\quad y_1 = x^{-1}\cos x$

21. $y'' + y'^3 \sin y = 0$

22. $(1 - x^2)y'' - 2xy' + 2y = 0,\quad y_1 = x$

23. (Motion) A small body moves on a straight line. Its velocity equals twice the reciprocal of its acceleration. If at $t = 0$ the body has distance 1 m from the origin and velocity 2 m/sec, what are its distance and velocity after 3 sec?

24. (Hanging cable) It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called a *catenary* (from Latin *catena* = the chain). Find and graph $y(x)$, assuming $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.

25. (Curves) Find and sketch or graph the curves passing through the origin with slope 1 for which the second derivative is proportional to the first.

26. WRITING PROJECT. General Properties of Solutions of Linear ODEs. Write a short essay (with proofs and simple examples of your own) that includes the following.

(a) The superposition principle.

(b) $y \equiv 0$ is a solution of the homogeneous equation (2) (called the **trivial solution**).

(c) The sum $y = y_1 + y_2$ of a solution y_1 of (1) and y_2 of (2) is a solution of (1).

(d) Explore possibilities of making further general statements on solutions of (1) and (2) (sums, differences, multiples).

27. CAS PROJECT. Linear Independence. Write a program for testing linear independence and dependence. Try it out on some of the problems in this problem set and on examples of your own.

2.2 Homogeneous Linear ODEs with Constant Coefficients

We shall now consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{1}$$

These equations have important applications, especially in connection with mechanical and electrical vibrations, as we shall see in Secs. 2.4, 2.8, and 2.9.

How to solve (1)? We remember from Sec. 1.5 that the solution of the *first-order* linear ODE with a constant coefficient k

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

$$y = e^{\lambda x}. \tag{2}$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

$$\lambda^2 + a\lambda + b = 0 \tag{3}$$

then the exponential function (2) is a solution of the ODE (1). Now from elementary algebra we recall that the roots of this quadratic equation (3) are

$$\lambda_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}), \qquad \lambda_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b}). \tag{4}$$

(3) and (4) will be basic because our derivation shows that the functions

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x} \tag{5}$$

are solutions of (1). Verify this by substituting (5) into (1).

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I) *Two real roots if* $a^2 - 4b > 0$,
(Case II) *A real double root if* $a^2 - 4b = 0$,
(Case III) *Complex conjugate roots if* $a^2 - 4b < 0$.

Case I. Two Distinct Real Roots λ_1 and λ_2

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant. The corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \tag{6}$$

EXAMPLE 1 General Solution in the Case of Distinct Real Roots

We can now solve $y'' - y = 0$ in Example 6 of Sec. 2.1 systematically. The characteristic equation is $\lambda^2 - 1 = 0$. Its roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence a basis of solutions is e^x and e^{-x} and gives the same general solution as before,

$$y = c_1 e^x + c_2 e^{-x}.$$ ■

EXAMPLE 2 Initial Value Problem in the Case of Distinct Real Roots

Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution. ***Step 1. General solution.*** The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are

$$\lambda_1 = \tfrac{1}{2}(-1 + \sqrt{9}) = 1 \qquad \text{and} \qquad \lambda_2 = \tfrac{1}{2}(-1 - \sqrt{9}) = -2$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular solution. Since $y'(x) = c_1 e^x - 2c_2 e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$
$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the *answer* $y = e^x + 3e^{-2x}$. Figure 29 shows that the curve begins at $y = 4$ with a negative slope (-5, but note that the axes have different scales!), in agreement with the initial conditions. ■

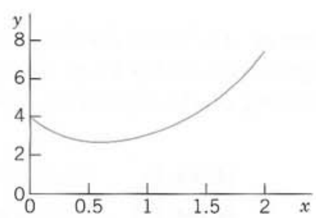


Fig. 29. Solution in Example 2

Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}.$$

To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$. Substituting this and its derivatives $y_2' = u'y_1 + uy_1'$ and y_2'' into (1), we first have

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0.$$

Collecting terms in u'', u', and u, as in the last section, we obtain

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0.$$

The expression in the last parentheses is zero, since y_1 is a solution of (1). The expression in the first parentheses is zero, too, since

$$2y_1' = -ae^{-ax/2} = -ay_1.$$

We are thus left with $u''y_1 = 0$. Hence $u'' = 0$. By two integrations, $u = c_1x + c_2$. To get a second independent solution $y_2 = uy_1$, we can simply choose $c_1 = 1$, $c_2 = 0$ and take $u = x$. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis. Hence in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2}, \qquad xe^{-ax/2}.$$

The corresponding general solution is

$$y = (c_1 + c_2x)e^{-ax/2}. \tag{7}$$

Warning. If λ is a *simple* root of (4), then $(c_1 + c_2x)e^{\lambda x}$ with $c_2 \neq 0$ is ***not*** a solution of (1).

EXAMPLE 3 General Solution in the Case of a Double Root

The characteristic equation of the ODE $y'' + 6y' + 9y = 0$ is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. It has the double root $\lambda = -3$. Hence a basis is e^{-3x} and xe^{-3x}. The corresponding general solution is $y = (c_1 + c_2x)e^{-3x}$. ■

EXAMPLE 4 Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Solution. The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2x)e^{-0.5x}.$$

We need its derivative

$$y' = c_2e^{-0.5x} - 0.5(c_1 + c_2x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0, \qquad y'(0) = c_2 - 0.5c_1 = -3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$. See Fig. 30. ■

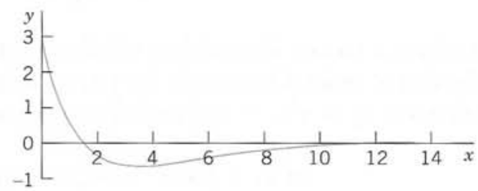


Fig. 30. Solution in Example 4

Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) and thus the solutions of the ODE (1) come at first out complex. However, we show that from them we can obtain a basis of *real* solutions

$$(8) \qquad y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x \qquad (\omega > 0)$$

where $\omega^2 = b - \frac{1}{4}a^2$. It can be verified by substitution that these are solutions in the present case. We shall derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient $\cot \omega x$ is not constant. Hence a real general solution in Case III is

$$\textbf{(9)} \qquad y = e^{-ax/2} (A \cos \omega x + B \sin \omega x) \qquad (A, B \text{ arbitrary}).$$

EXAMPLE 5 Complex Roots. Initial Value Problem

Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution. ***Step 1. General solution.*** The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x} \sin 3x$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x).$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

$$y = e^{-0.2x} \sin 3x.$$

Figure 31 shows y and the curves of $e^{-0.2x}$ and $-e^{-0.2x}$ (dashed), between which the curve of y oscillates. Such "damped vibrations" (with $x = t$ being time) have important mechanical and electrical applications, as we shall soon see (in Sec. 2.4). ■

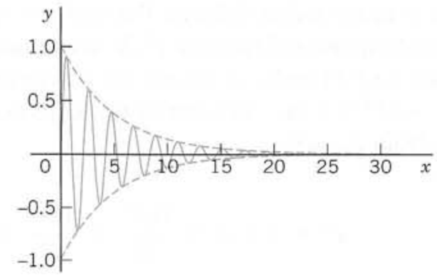


Fig. 31. Solution in Example 5

EXAMPLE 6 Complex Roots

A general solution of the ODE

$$y'' + \omega^2 y = 0 \qquad (\omega \text{ constant, not zero})$$

is

$$y = A \cos \omega x + B \sin \omega x.$$

With $\omega = 1$ this confirms Example 4 in Sec. 2.1. ■

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

It is very interesting that in applications to mechanical systems or electrical circuits, these three cases correspond to three different forms of motion or flows of current, respectively. We shall discuss this basic relation between theory and practice in detail in Sec. 2.4 (and again in Sec. 2.8).

Derivation in Case III. Complex Exponential Function

If verification of the solutions in (8) satisfies you, skip the systematic derivation of these real solutions from the complex solutions by means of the complex exponential function e^z of a complex variable $z = r + it$. We write $r + it$, not $x + iy$ because x and y occur in the ODE. The definition of e^z in terms of the real functions e^r, $\cos t$, and $\sin t$ is

$$e^z = e^{r+it} = e^r e^{it} = e^r(\cos t + i \sin t). \tag{10}$$

This is motivated as follows. For real $z = r$, hence $t = 0$, $\cos 0 = 1$, $\sin 0 = 0$, we get the real exponential function e^r. It can be shown that $e^{z_1+z_2} = e^{z_1}e^{z_2}$, just as in real. (Proof in Sec. 13.5.) Finally, if we use the Maclaurin series of e^z with $z = it$ as well as $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc., and reorder the terms as shown (this is permissible, as can be proved), we obtain the series

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\ &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \cdots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \cdots\right) \\ &= \cos t + i \sin t. \end{aligned}$$

(Look up these real series in your calculus book if necessary.) We see that we have obtained the formula

$$e^{it} = \cos t + i \sin t, \tag{11}$$

called the **Euler formula.** Multiplication by e^r gives (10).

For later use we note that $e^{-it} = \cos(-t) + i \sin(-t) = \cos t - i \sin t$, so that by addition and subtraction of this and (11),

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}). \tag{12}$$

After these comments on the definition (10), let us now turn to Case III.

In Case III the radicand $a^2 - 4b$ in (4) is negative. Hence $4b - a^2$ is positive and, using $\sqrt{-1} = i$, we obtain in (4)

$$\tfrac{1}{2}\sqrt{a^2 - 4b} = \tfrac{1}{2}\sqrt{-(4b - a^2)} = \sqrt{-(b - \tfrac{1}{4}a^2)} = i\sqrt{b - \tfrac{1}{4}a^2} = i\omega$$

with ω defined as in (8). Hence in (4),

$$\lambda_1 = \tfrac{1}{2}a + i\omega \qquad \text{and, similarly,} \qquad \lambda_2 = \tfrac{1}{2}a - i\omega.$$

Using (10) with $r = -\frac{1}{2}ax$ and $t = \omega x$, we thus obtain

$$\begin{aligned} e^{\lambda_1 x} &= e^{-(a/2)x + i\omega x} = e^{-(a/2)x}(\cos \omega x + i \sin \omega x) \\ e^{\lambda_2 x} &= e^{-(a/2)x - i\omega x} = e^{-(a/2)x}(\cos \omega x - i \sin \omega x). \end{aligned}$$

We now add these two lines and multiply the result by $\frac{1}{2}$. This gives y_1 as in (8). Then we subtract the second line from the first and multiply the result by $1/(2i)$. This gives y_2 as in (8). These results obtained by addition and multiplication by constants are again solutions, as follows from the superposition principle in Sec. 2.1. This concludes the derivation of these real solutions in Case III.

PROBLEM SET 2.2

1–14 GENERAL SOLUTION

Find a general solution. Check your answer by substitution.

1. $y'' - 6y' - 7y = 0$

2. $10y'' - 7y' + 1.2y = 0$

3. $4y'' - 20y' + 25y = 0$

4. $y'' + 4\pi y' + 4\pi^2 y = 0$

5. $100y'' + 20y' - 99y = 0$

6. $y'' + 2y' + 5y = 0$

7. $y'' - y' + 2.5y = 0$

8. $y'' + 2.6y' + 1.69y = 0$

9. $y'' - 2y' - 5.25y = 0$

10. $y'' - 2y = 0$

11. $y'' + 9\pi^2 y = 0$

12. $y'' + 2.4y' + 4.0y = 0$

13. $y'' - 144y = 0$

14. $y'' + y' - 0.96y = 0$

15–20 FIND ODE

Find an ODE $y'' + ay' + by = 0$ for the given basis.

15. e^{2x}, e^{x}

16. $e^{0.5x}, e^{-3.5x}$

17. $e^{x\sqrt{3}}, xe^{x\sqrt{3}}$

18. $1, e^{-3x}$

19. e^{4x}, e^{-4x}

20. $e^{(-1+i)x}, e^{-(1+i)x}$

21–32 INITIAL VALUE PROBLEMS

Solve the initial value problem. Check that your answer satisfies the ODE as well as the initial conditions. (Show the details of your work.)

21. $y'' - 2y' - 3y = 0,\ y(0) = 2,\ y'(0) = 14$

22. $y'' + 2y' + y = 0,\ y(0) = 4,\ y'(0) = -6$

23. $y'' + 4y' + 5y = 0,\ y(0) = 2,\ y'(0) = -5$

24. $10y'' - 50y' + 65y = 0,\ y(0) = 1.5,\ y'(0) = 1.5$

25. $y'' + \pi y' = 0,\ y(0) = 3,\ y'(0) = -\pi$

26. $10y'' + 18y' + 5.6y = 0,\ y(0) = 4,\ y'(0) = -3.8$

27. $10y'' + 5y' + 0.625y = 0,\ y(0) = 2,\ y'(0) = -4.5$

28. $y'' - 9y = 0,\ y(0) = -2,\ y'(0) = -12$

29. $20y'' + 4y' + y = 0,\ y(0) = 3.2,\ y'(0) = 0$

30. $y'' + 2ky' + (k^2 + \omega^2)y = 0,\ y(0) = 1,\ y'(0) = -k$

31. $y'' - 25y = 0,\ y(0) = 0,\ y'(0) = 40$

32. $y'' - 2y' - 24y = 0,\ y(0) = 0,\ y'(0) = 20$

33. **(Instability)** Solve $y'' - y = 0$ for the initial conditions $y(0) = 1, y'(0) = -1$. Then change the initial conditions to $y(0) = 1.001$, $y'(0) = -0.999$ and explain why this small change of 0.001 at $x = 0$ causes a large change later, e.g., 22 at $x = 10$.

34. **TEAM PROJECT. General Properties of Solutions**

(A) Coefficient formulas. Show how a and b in (1) can be expressed in terms of λ_1 and λ_2. Explain how these formulas can be used in constructing equations for given bases.

(B) Root zero. Solve $y'' + 4y' = 0$ (i) by the present method, and (ii) by reduction to first order. Can you explain why the result must be the same in both cases? Can you do the same for a general ODE $y'' + ay' = 0$?

(C) Double root. Verify directly that $xe^{\lambda x}$ with $\lambda = -a/2$ is a solution of (1) in the case of a double root. Verify and explain why $y = e^{-2x}$ is a solution of $y'' - y' - 6y = 0$ but xe^{-2x} is not.

(D) Limits. Double roots should be limiting cases of distinct roots λ_1, λ_2 as, say, $\lambda_2 \to \lambda_1$. Experiment with this idea. (Remember l'Hôpital's rule from calculus.) Can you arrive at $xe^{\lambda_1 x}$? Give it a try.

35. **(Verification)** Show by substitution that y_1 in (8) is a solution of (1).

2.3 Differential Operators. *Optional*

This short section can be omitted without interrupting the flow of ideas; it will not be used in the sequel (except for the notations Dy, D^2y, etc., for y', y'', etc.).

Operational calculus means the technique and application of operators. Here, an **operator** is a transformation that transforms a function into another function. Hence differential calculus involves an operator, the **differential operator** D, which transforms a (differentiable) function into its derivative. In operator notation we write

$$Dy = y' = \frac{dy}{dx}. \qquad (1)$$

Similarly, for the higher derivatives we write $D^2y = D(Dy) = y''$, and so on. For example, $D \sin = \cos$, $D^2 \sin = -\sin$, etc.

For a homogeneous linear ODE $y'' + ay' + by = 0$ with constant coefficients we can now introduce the **second-order differential operator**

$$L = P(D) = D^2 + aD + bI,$$

where I is the **identity operator** defined by $Iy = y$. Then we can write that ODE as

$$Ly = P(D)y = (D^2 + aD + bI)y = 0. \tag{2}$$

P suggests "polynomial." L is a **linear operator.** By definition this means that if Ly and Lw exist (this is the case if y and w are twice differentiable), then $L(cy + kw)$ exists for any constants c and k, and

$$L(cy + kw) = cLy + kLw.$$

Let us show that from (2) we reach agreement with the results in Sec. 2.2. Since $(De^{\lambda})(x) = \lambda e^{\lambda x}$ and $(D^2e^{\lambda})(x) = \lambda^2 e^{\lambda x}$, we obtain

$$\begin{aligned} Le^{\lambda}(x) &= P(D)e^{\lambda}(x) = (D^2 + aD + bI)e^{\lambda}(x) \\ &= (\lambda^2 + a\lambda + b)e^{\lambda x} = P(\lambda)e^{\lambda x} = 0. \end{aligned} \tag{3}$$

This confirms our result of Sec. 2.2 that $e^{\lambda x}$ *is a solution of the ODE* (2) *if and only if* λ *is a solution of the characteristic equation* $P(\lambda) = 0$.

$P(\lambda)$ is a polynomial in the usual sense of algebra. If we replace λ by the operator D, we obtain the "operator polynomial" $P(D)$. *The point of this operational calculus is that* $P(D)$ *can be treated just like an algebraic quantity.* In particular, we can factor it.

EXAMPLE 1 Factorization, Solution of an ODE

Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$.

Solution. $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $I^2 = I$. Now $(D - 8I)y = y' - 8y = 0$ has the solution $y_1 = e^{8x}$. Similarly, the solution of $(D + 5I)y = 0$ is $y_2 = e^{-5x}$. This is a basis of $P(D)y = 0$ on any interval. From the factorization we obtain the ODE, as expected,

$$\begin{aligned} (D - 8I)(D + 5I)y &= (D - 8I)(y' + 5y) = D(y' + 5y) - 8(y' + 5y) \\ &= y'' + 5y' - 8y' - 40y = y'' - 3y' - 40y = 0. \end{aligned}$$

Verify that this agrees with the result of our method in Sec. 2.2. This is not unexpected because we factored $P(D)$ in the same way as the characteristic polynomial $P(\lambda) = \lambda^2 - 3\lambda - 40$. ■

It was essential that L in (2) has *constant* coefficients. Extension of operator methods to variable-coefficient ODEs is more difficult and will not be considered here.

If operational methods were limited to the simple situations illustrated in this section, it would perhaps not be worth mentioning. Actually, the power of the operator approach appears in more complicated engineering problems, as we shall see in Chap. 6.

PROBLEM SET 2.3

1–5 APPLICATION OF DIFFERENTIAL OPERATORS

Apply the given operator to the given functions. (Show all steps in detail.)

1. $(D - I)^2$; e^x, xe^x, $\sin x$
2. $8D^2 + 2D - I$; $\cosh \frac{1}{2}x$, $\sinh \frac{1}{2}x$, $e^{x/2}$
3. $D - 0.4I$; $2x^3 - 1$, $e^{0.4x}$, $xe^{0.4x}$
4. $(D + 5I)(D - I)$; $e^{-5x} \sin x$, e^{5x}, x^2
5. $(D - 4I)(D + 3I)$; $x^3 - x^2$, $\sin 4x$, e^{-3x}

6–13 GENERAL SOLUTION

Factor as in the text and solve. (Show the details.)

6. $(D^2 - 5.5D + 6.66I)y = 0$
7. $(D + 2I)^2 y = 0$
8. $(D^2 - 0.49I)y = 0$
9. $(D^2 + 6D + 13I)y = 0$
10. $(10D^2 + 2D + 1.7I)y = 0$
11. $(D^2 + 4.1D + 3.1I)y = 0$
12. $(4D^2 + 4\pi D + \pi^2 I)y = 0$
13. $(D^2 + 17.64\omega^2 I)y = 0$

14. **(Double root)** If $D^2 + aD + bI$ has distinct roots μ and λ, show that a particular solution is $y = (e^{\mu x} - e^{\lambda x})/(\mu - \lambda)$. Obtain from this a solution $xe^{\lambda x}$ by letting $\mu \to \lambda$ and applying l'Hôpital's rule.
15. **(Linear operator)** Illustrate the linearity of L in (2) by taking $c = 4$, $k = -6$, $y = e^{2x}$, and $w = \cos 2x$. Prove that L is linear.
16. **(Definition of linearity)** Show that the definition of linearity in the text is equivalent to the following. If $L[y]$ and $L[w]$ exist, then $L[y + w]$ exists and $L[cy]$ and $L[kw]$ exist for all constants c and k, and $L[y + w] = L[y] + L[w]$ as well as $L[cy] = cL[y]$ and $L[kw] = kL[w]$.

2.4 Modeling: Free Oscillations (Mass–Spring System)

Linear ODEs with constant coefficients have important applications in mechanics, as we show now (and in Sec. 2.8), and in electric circuits (to be shown in Sec. 2.9). In this section we consider a basic mechanical system, a mass on an elastic spring ("mass-spring system," Fig. 32), which moves up and down. Its model will be a homogeneous linear ODE.

Setting Up the Model

We take an ordinary spring that resists compression as well extension and suspend it vertically from a fixed support, as shown in Fig. 32. At the lower end of the spring we

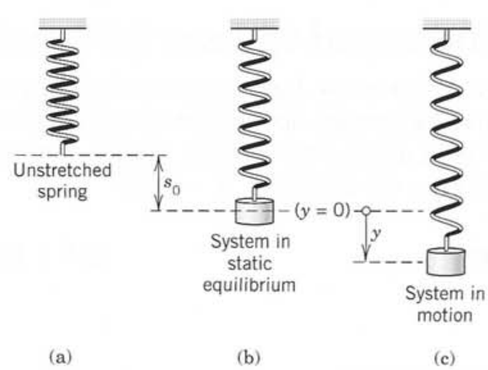


Fig. 32. Mechanical mass–spring system

attach a body of mass m. We assume m to be so large that we can neglect the mass of the spring. If we pull the body down a certain distance and then release it, it starts moving. We assume that it moves strictly vertically.

How can we obtain the motion of the body, say, the displacement $y(t)$ as function of time t? Now this motion is determined by **Newton's second law**

$$\text{Mass} \times \text{Acceleration} = my'' = \text{Force} \tag{1}$$

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the body.

(For systems of units and conversion factors, see the inside of the front cover.)

We choose the ***downward direction as the positive direction,*** thus regarding downward forces as positive and upward forces as negative.

Consider Fig. 32. The spring is first unstretched. We now attach the body. This stretches the spring by an amount s_0 shown in the figure. It causes an upward force F_0 in the spring. Experiments show that F_0 is proportional to the stretch s_0, say,

$$F_0 = -ks_0 \qquad \textbf{(Hooke's law}^2\textbf{)}. \tag{2}$$

k (> 0) is called the **spring constant** (or *spring modulus*). The minus sign indicates that F_0 points upward, in our negative direction. Stiff springs have large k. (Explain!)

The extension s_0 is such that F_0 in the spring balances the weight $W = mg$ of the body (where $g = 980 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2$ is the gravitational constant). Hence $F_0 + W = -ks_0 + mg = 0$. These forces will not affect the motion. Spring and body are again at rest. This is called the **static equilibrium** of the system (Fig. 32b). We measure the displacement $y(t)$ of the body from this 'equilibrium point' as the origin $y = 0$, downward positive and upward negative.

From the position $y = 0$ we pull the body downward. This further stretches the spring by some amount $y > 0$ (the distance we pull it down). By Hooke's law this causes an (additional) upward force F_1 in the spring,

$$F_1 = -ky.$$

F_1 is a **restoring force.** It has the tendency to *restore* the system, that is, to pull the body back to $y = 0$.

Undamped System: ODE and Solution

Every system has damping—otherwise it would keep moving forever. But practically, the effect of damping may often be negligible, for example, for the motion of an iron ball on a spring during a few minutes. Then F_1 is the only force in (1) causing the motion. Hence (1) gives the model $my'' = -ky$ or

$$my'' + ky = 0. \tag{3}$$

---

[2]ROBERT HOOKE (1635–1703), English physicist, a forerunner of Newton with respect to the law of gravitation.

By the method in Sec. 2.2 (see Example 6) we obtain as a general solution

$$(4) \qquad y(t) = A \cos \omega_0 t + B \sin \omega_0 t, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

The corresponding motion is called a **harmonic oscillation.**

Since the trigonometric functions in (4) have the period $2\pi/\omega_0$, the body executes $\omega_0/2\pi$ cycles per second. This is the **frequency** of the oscillation, which is also called the **natural frequency** of the system. It is measured in cycles per second. Another name for cycles/sec is hertz (Hz).[3]

The sum in (4) can be combined into a phase-shifted cosine with amplitude $C = \sqrt{A^2 + B^2}$ and phase angle $\delta = \arctan (B/A)$,

$$(4^*) \qquad y(t) = C \cos (\omega_0 t - \delta).$$

To verify this, apply the addition formula for the cosine [(6) in App. 3.1] to (4*) and then compare with (4). Equation (4) is simpler in connection with initial value problems, whereas (4*) is physically more informative because it exhibits the amplitude and phase of the oscillation.

Figure 33 shows typical forms of (4) and (4*), all corresponding to some positive initial displacement $y(0)$ [which determines $A = y(0)$ in (4)] and different initial velocities $y'(0)$ [which determine $B = y'(0)/\omega_0$].

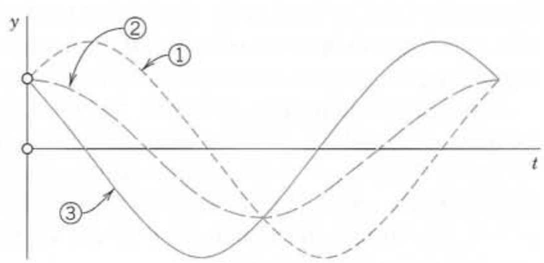


Fig. 33. Harmonic oscillations

EXAMPLE 1 Undamped Motion. Harmonic Oscillation

If an iron ball of weight $W = 98$ nt (about 22 lb) stretches a spring 1.09 m (about 43 in.), how many cycles per minute will this mass–spring system execute? What will its motion be if we pull down the weight an additional 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke's law (2) with W as the force and 1.09 meter as the stretch gives $W = 1.09k$; thus $k = W/1.09 = 98/1.09 = 90$ [kg/sec^2] $= 90$ [nt/meter]. The mass is $m = W/g = 98/9.8 = 10$ [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] $= 29$ [cycles/min].

[3]HEINRICH HERTZ (1857–1894), German physicist, who discovered electromagnetic waves, as the basis of wireless communication developed by GUGLIELMO MARCONI (1874–1937), Italian physicist (Nobel prize in 1909).

From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16 \cos 3t \text{ [meter]} \qquad \text{or} \qquad 0.52 \cos 3t \text{ [ft]} \qquad \text{(Fig. 34).}$$

If you have a chance of experimenting with a mass–spring system, don't miss it. You will be surprised about the good agreement between theory and experiment, usually within a fraction of one percent if you measure carefully. ■

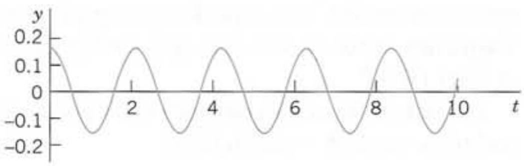


Fig. 34. Harmonic oscillation in Example 1

Damped System: ODE and Solutions

We now add a **damping force**

$$F_2 = -cy'$$

to our model $my'' = -ky$, so that we have $my'' = -ky - cy'$ or

$$\textbf{(5)} \qquad my'' + cy' + ky = 0.$$

Physically this can be done by connecting the body to a dashpot; see Fig. 35. We assume this new force to be proportional to the velocity $y' = dy/dt$, as shown. This is generally a good approximation, at least for small velocities.

c is called the **damping constant.** We show that c is positive. If at some instant, y' is positive, the body is moving downward (which is the positive direction). Hence the damping force $F_2 = -cy'$, always acting *against* the direction of motion, must be an upward force, which means that it must be negative, $F_2 = -cy' < 0$, so that $-c < 0$ and $c > 0$. For an upward motion, $y' < 0$ and we have a downward $F_2 = -cy > 0$; hence $-c < 0$ and $c > 0$, as before.

The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by m)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

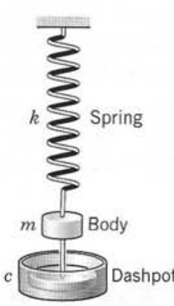


Fig. 35. Damped system

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$\text{(6)}\quad \lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

It is now most interesting that depending on the amount of damping (much, medium, or little) there will be three types of motion corresponding to the three Cases I, II, II in Sec. 2.2:

Case I.	$c^2 > 4mk.$	*Distinct real roots* $\lambda_1, \lambda_2.$	**(Overdamping)**
Case II.	$c^2 = 4mk.$	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk.$	*Complex conjugate roots.*	**(Underdamping)**

Discussion of the Three Cases

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots. In this case the corresponding general solution of (5) is

$$\text{(7)}\qquad y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Figure 36 shows (7) for some typical initial conditions.

Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$,

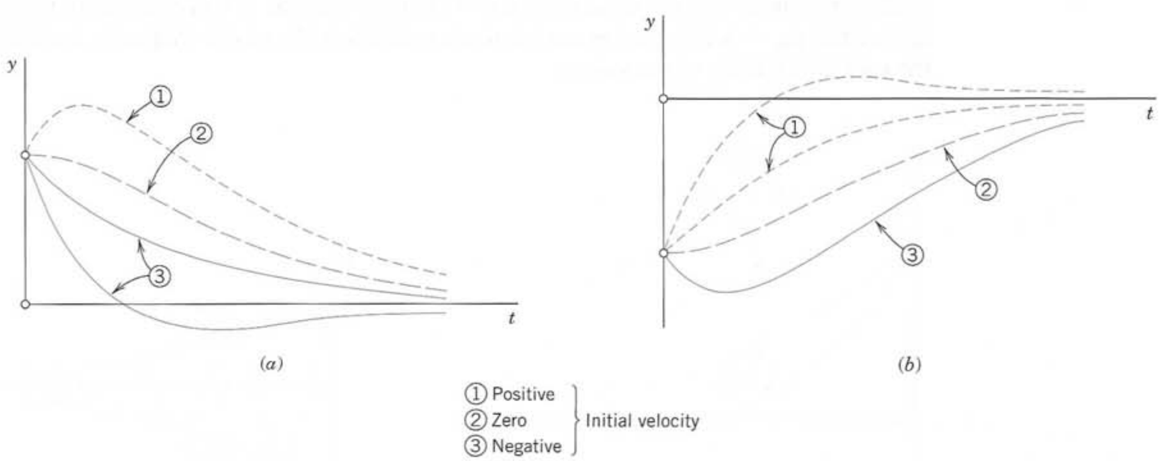


Fig. 36. Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{8}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero. If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. Figure 37 shows typical forms of (8). Note that they look almost like those in the previous figure.

Case III. Underdamping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0). \tag{9}$$

(We write ω^* to reserve ω for driving and electromotive forces in Secs. 2.8 and 2.9.) The roots of the characteristic equation are now complex conjugate,

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta) \tag{10}$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

This represents **damped oscillations.** Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 38, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which $\cos (\omega^* t - \delta)$ equals 1 or -1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (>0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

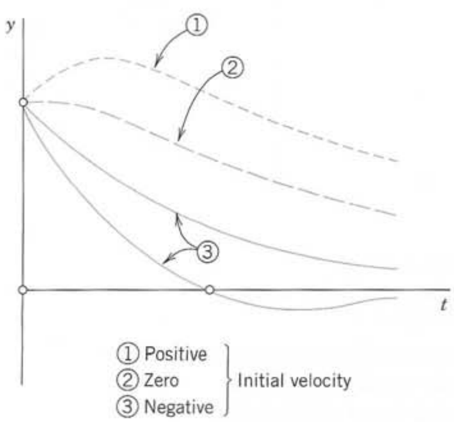


Fig. 37. Critical damping [see (8)]

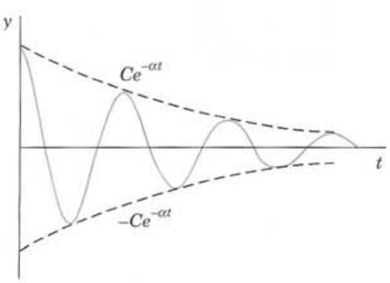


Fig. 38. Damped oscillation in Case III [see (10)]

EXAMPLE 2 The Three Cases of Damped Motion

How does the motion in Example 1 change if we change the damping constant c to one of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

$$\text{(I) } c = 100 \text{ kg/sec}, \qquad \text{(II) } c = 60 \text{ kg/sec}, \qquad \text{(III) } c = 10 \text{ kg/sec}.$$

Solution. It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

(I) With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$. It has the roots -9 and -1. This gives the general solution

$$y = c_1e^{-9t} + c_2e^{-t}. \qquad \text{We also need} \qquad y' = -9c_1e^{-9t} - c_2e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

It approaches 0 as $t \to \infty$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root -3. Hence the corresponding general solution is

$$y = (c_1 + c_2t)e^{-3t}. \qquad \text{We also need} \qquad y' = (c_2 - 3c_1 - 3c_2t)e^{-3t}.$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

(III) The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical c, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

This gives the general solution

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 39. ■

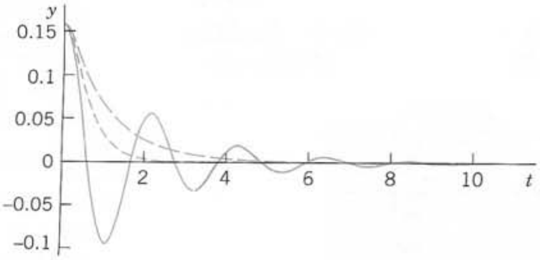


Fig. 39. The three solutions in Example 2

This section concerned ***free motions*** of mass-spring systems. Their models are *homogeneous* linear ODEs. *Nonhomogeneous* linear ODEs will arise as models of **forced motions,** that is, motions under the influence of a "driving force". We shall study them in Sec. 2.8, after we have learned how to solve those ODEs.

PROBLEM SET 2.4

1–8 MOTION WITHOUT DAMPING (HARMONIC OSCILLATIONS)

1. **(Initial value problem)** Find the harmonic motion (4) that starts from y_0 with initial velocity v_0. Graph or sketch the solutions for $\omega_0 = \pi$, $y_0 = 1$, and various v_0 of your choice on common axes. At what t-values do all these curves intersect? Why?
2. **(Spring combinations)** Find the frequency of vibration of a ball of mass $m = 3$ kg on a spring of modulus (i) $k_1 = 27$ nt/m, (ii) $k_2 = 75$ nt/m, (iii) on these springs in parallel (see Fig. 40), (iv) in series, that is, the ball hangs on one spring, which in turn hangs on the other spring.
3. **(Pendulum)** Find the frequency of oscillation of a pendulum of length L (Fig. 41), neglecting air resistance and the weight of the rod, and assuming θ to be so small that $\sin \theta$ practically equals θ.
4. **(Frequency)** What is the frequency of a harmonic oscillation if the static equilibrium position of the ball is 10 cm lower than the lower end of the spring before the ball is attached?
5. **(Initial velocity)** Could you make a harmonic oscillation move faster by giving the body a greater initial push?
6. **(Archimedian principle)** This principle states that the buoyancy force equals the weight of the water displaced by the body (partly or totally submerged). The cylindrical buoy of diameter 60 cm in Fig. 42 is floating in water with its axis vertical. When depressed downward in the water and released, it vibrates with period 2 sec. What is its weight?

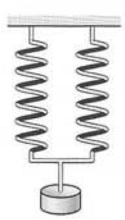


Fig. 40. Parallel springs (Problem 2)

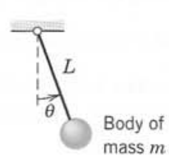


Fig. 41. Pendulum (Problem 3)

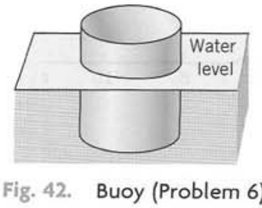


Fig. 42. Buoy (Problem 6)

7. **(Frequency)** How does the frequency of a harmonic motion change if we take (i) a spring of three times the modulus, (ii) a heavier ball?
8. **TEAM PROJECT. Harmonic Motions in Different Physical Systems.** Different physical or other systems may have the same or similar models, thus showing the ***unifying power of mathematical methods.*** Illustrate this for the systems in Figs. 43–45.

   (a) Flat spring (Fig. 43). The spring is horizontally clamped at one end, and a body of weight 25 nt (about 5.6 lb) is attached at the other end. Find the motion of the system, assuming that its static equilibrium is 2 cm below the horizontal line, we let the system start from this position with initial velocity 15 cm/sec, and damping is negligible.

   (b) Torsional vibrations (Fig. 44). Undamped torsional vibrations (rotations back and forth) of a wheel attached to an elastic thin rod are modeled by the ODE $I_0\theta'' + K\theta = 0$, where θ is the angle measured from the state of equilibrium, I_0 is the polar moment of inertia of the wheel about its center, and K is the torsional stiffness of the rod. Solve this ODE for $K/I_0 = 17.64 \text{ sec}^{-2}$, initial angle 45°, and initial angular velocity $15° \text{ sec}^{-1}$.

   (c) Water in a tube (Fig. 45). What is the frequency of vibration of 5 liters of water (about 1.3 gal) in a U-shaped tube of diameter 4 cm, neglecting friction?

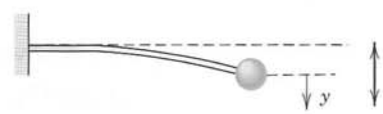


Fig. 43. Flat spring (Project 8a)

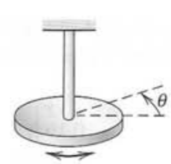


Fig. 44. Torsional vibrations (Project 8b)

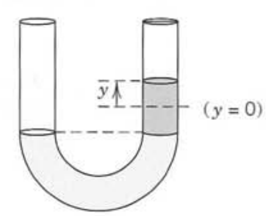


Fig. 45. Tube (Project 8c)

9–17 DAMPED MOTION

9. **(Frequency)** Find an approximation formula for ω^* in terms of ω_0 by applying the binomial theorem in (9) and retaining only the first two terms. How good is the approximation in Example 2, III?

10. (Extrema) Find the location of the maxima and minima of $y = e^{-2t} \cos t$ obtained approximately from a graph of y and compare it with the exact values obtained by calculation.

11. (Maxima) Show that the maxima of an underdamped motion occur at equidistant t-values and find the distance.

12. (Logarithmic decrement) Show that the ratio of two consecutive maximum amplitudes of a damped oscillation (10) is constant, and the natural logarithm of this ratio, called the *logarithmic decrement,* equals $\Delta = 2\pi\alpha/\omega^*$. Find Δ for the solutions of $y'' + 2y' + 5y = 0$.

13. (Shock absorber) What is the smallest value of the damping constant of a shock absorber in the suspension of a wheel of a car (consisting of a spring and an absorber) that will provide (theoretically) an oscillation-free ride if the mass of the car is 2000 kg and the spring constant equals 4500 kg/sec^2?

14. (Damping constant) Consider an underdamped motion of a body of mass $m = 2$ kg. If the time between two consecutive maxima is 2 sec and the maximum amplitude decreases to $\frac{1}{4}$ of its initial value after 15 cycles, what is the damping constant of the system?

15. (Initial value problem) Find the critical motion (8) that starts from y_0 with initial velocity v_0. Graph solution curves for $\alpha = 1$, $y_0 = 1$ and several v_0 such that (i) the curve does not intersect the t-axis, (ii) it intersects it at $t = 1, 2, \cdots, 5$, respectively.

16. (Initial value problem) Find the overdamped motion (7) that starts from y_0 with initial velocity v_0.

17. (Overdamping) Show that in the overdamped case, the body can pass through $y = 0$ at most once.

18. CAS PROJECT. Transition Between Cases I, II, III. Study this transition in terms of graphs of typical solutions. (Cf. Fig. 46.)

(a) ***Avoiding unnecessary generality is part of good modeling.*** Decide that the initial value problems (A) and (B),

(A) $\quad y'' + cy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$

(B) the same with different c and $y'(0) = -2$ (instead of 0), will give practically as much information as a problem with other m, k, $y(0)$, $y'(0)$.

(b) ***Consider*** **(A).** Choose suitable values of c, perhaps better ones than in Fig. 46 for the transition from Case III to II and I. Guess c for the curves in the figure.

(c) ***Time to go to rest.*** Theoretically, this time is infinite (why?). Practically, the system is at rest when its motion has become very small, say, less than 0.1% of the initial displacement (this choice being up to us), that is in our case,

(11) $|y(t)| < 0.001$ for all t greater than some t_1.

In engineering constructions, damping can often be varied without too much trouble. Experimenting with your graphs, find empirically a relation between t_1 and c.

(d) ***Solve*** **(A)** ***analytically.*** Give a reason why the solution c of $y(t_2) = -0.001$, with t_2 the solution of $y'(t) = 0$, will give you the best possible c satisfying (11).

(e) Consider (B) empirically as in (a) and (b). What is the main difference between (B) and (A)?

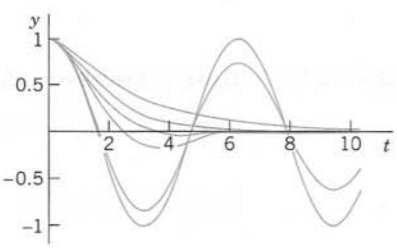


Fig. 46. CAS Project 18

2.5 Euler–Cauchy Equations

Euler–Cauchy equations[4] are ODEs of the form

$$x^2y'' + axy' + by = 0 \tag{1}$$

[4]LEONHARD EULER (1707–1783) was an enormously creative Swiss mathematician. He made fundamental contributions to almost all branches of mathematics and its application to physics. His important books on algebra and calculus contain numerous basic results of his own research. The great French mathematician AUGUSTIN LOUIS CAUCHY (1789–1857) is the father of modern analysis. He is the creator of complex analysis and had great influence on ODEs, PDEs, infinite series, elasticity theory, and optics.

with given constants a and b and unknown $y(x)$. We substitute

$$y = x^m \tag{2}$$

and its derivatives $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ into (1). This gives

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0.$$

We now see that (2) was a rather natural choice because we have obtained a common factor x^m. Dropping it, we have the auxiliary equation $m(m-1) + am + b = 0$ or

$$m^2 + (a-1)m + b = 0. \qquad \text{(Note: } a - 1\text{, not } a.) \tag{3}$$

Hence $y = x^m$ is a solution of (1) if and only if m is a root of (3). The roots of (3) are

$$m_1 = \tfrac{1}{2}(1-a) + \sqrt{\tfrac{1}{4}(1-a)^2 - b}, \qquad m_2 = \tfrac{1}{2}(1-a) - \sqrt{\tfrac{1}{4}(1-a)^2 - b}. \tag{4}$$

Case I. If the roots m_1 and m_2 are real and different, then solutions are

$$y_1(x) = x^{m_1} \qquad \text{and} \qquad y_2(x) = x^{m_2}$$

They are linearly independent since their quotient is not constant. Hence they constitute a basis of solutions of (1) for all x for which they are real. The corresponding general solution for all these x is

$$y = c_1 x^{m_1} + c_2 x^{m_2} \qquad (c_1, c_2 \text{ arbitrary}). \tag{5}$$

EXAMPLE 1 General Solution in the Case of Different Real Roots

The Euler–Cauchy equation

$$x^2 y'' + 1.5xy' - 0.5y = 0$$

has the auxiliary equation

$$m^2 + 0.5m - 0.5 = 0. \qquad \text{(Note: 0.5, not 1.5!)}$$

The roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0).$$

Case II. Equation (4) shows that the auxiliary equation (3) has a double root $m_1 = \frac{1}{2}(1-a)$ if and only if $(1-a)^2 - 4b = 0$. The Euler–Cauchy equation (1) then has the form

$$x^2 y'' + axy' + \tfrac{1}{4}(1-a)^2 y = 0. \tag{6}$$

A solution is $y_1 = x^{(1-a)/2}$. To obtain a second linearly independent solution, we apply the method of reduction of order from Sec. 2.1 as follows. Starting from $y_2 = uy_1$, we obtain for u the expression (9), Sec. 2.1, namely,

$$u = \int U\,dx \qquad \text{where} \qquad U = \frac{1}{y_1^{\,2}} \exp\left(-\int p\,dx\right).$$

Here it is crucial that p is taken from the ODE written in standard form, in our case,

$$(6^*) \qquad y'' + \frac{a}{x}\,y' + \frac{(1-a)^2}{4x^2}\,y = 0.$$

This shows that $p = a/x$ (not ax). Hence its integral is $a \ln x = \ln (x^a)$, the exponential function in U is $1/x^a$, and division by $y_1{}^2 = x^{1-a}$ gives $U = 1/x$, and $u = \ln x$ by integration.

Thus, in this "critical case," a basis of solutions for positive x is $y_1 = x^m$ and $y_2 = x^m \ln x$, where $m = \frac{1}{2}(1 - a)$. Linear independence follows from the fact that the quotient of these solutions is not constant. Hence, for all x for which y_1 and y_2 are defined and real, a general solution is

$$(7) \qquad y = (c_1 + c_2 \ln x)\,x^m, \qquad m = \tfrac{1}{2}(1 - a).$$

EXAMPLE 2 General Solution in the Case of a Double Root

The Euler–Cauchy equation $x^2y'' - 5xy' + 9y = 0$ has the auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is

$$y = (c_1 + c_2 \ln x)x^3.$$

■

Case III. The case of complex roots is of minor practical importance, and it suffices to present an example that explains the derivation of real solutions from complex ones.

EXAMPLE 3 Real General Solution in the Case of Complex Roots

The Euler–Cauchy equation

$$x^2y'' + 0.6xy' + 16.04y = 0$$

has the auxiliary equation $m^2 - 0.4m + 16.04 = 0$. The roots are complex conjugate, $m_1 = 0.2 + 4i$ and $m_2 = 0.2 - 4i$, where $i = \sqrt{-1}$. (We know from algebra that if a polynomial with real coefficients has complex roots, these are always conjugate.) Now use the trick of writing $x = e^{\ln x}$ and obtain

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2}e^{(4 \ln x)\,i},$$

$$x^{m_2} = x^{0.2-4i} = x^{0.2}(e^{\ln x})^{-4i} = x^{0.2}e^{-(4 \ln x)\,i}.$$

Next apply Euler's formula (11) in Sec. 2.2 with $t = 4 \ln x$ to these two formulas. This gives

$$x^{m_1} = x^{0.2}[\cos (4 \ln x) + i \sin (4 \ln x)],$$

$$x^{m_2} = x^{0.2}[\cos (4 \ln x) - i \sin (4 \ln x)].$$

Add these two formulas, so that the sine drops out, and divide the result by 2. Then subtract the second formula from the first, so that the cosine drops out, and divide the result by $2i$. This yields

$$x^{0.2} \cos (4 \ln x) \qquad \text{and} \qquad x^{0.2} \sin (4 \ln x)$$

respectively. By the superposition principle in Sec. 2.2 these are solutions of the Euler–Cauchy equation (1). Since their quotient $\cot (4 \ln x)$ is not constant, they are linearly independent. Hence they form a basis of solutions, and the corresponding real general solution for all positive x is

$$(8) \qquad y = x^{0.2}[A \cos (4 \ln x) + B \sin (4 \ln x)].$$

Figure 47 shows typical solution curves in the three cases discussed, in particular the basis functions in Examples 1 and 3. ■

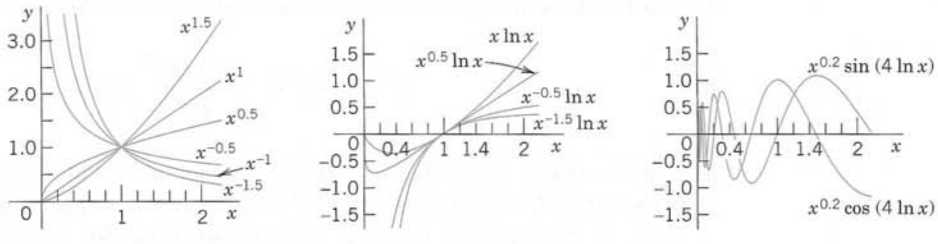


Fig. 47. Euler–Cauchy equations

EXAMPLE 4 Boundary Value Problem. Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

Physical Information. $v(r)$ is a solution of the Euler–Cauchy equation $rv'' + 2v' = 0$, where $v' = dv/dr$.

Solution. The auxiliary equation is $m^2 + m = 0$. It has the roots 0 and -1. This gives the general solution $v(r) = c_1 + c_2/r$. From the "boundary conditions" (the potentials on the spheres) we obtain

$$v(5) = c_1 + \frac{c_2}{5} = 110, \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$. *Answer:* $v(r) = -110 + 1100/r$ V. Figure 48 shows that the potential is not a straight line, as it would be for a potential between two parallel plates. For example, on the sphere of radius 7.5 cm it is not $110/2 = 55$ V, but considerably less. (What is it?) ■

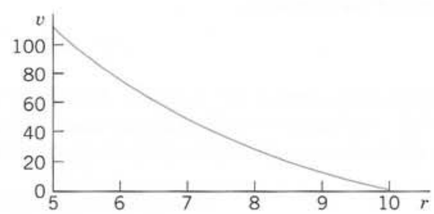


Fig. 48. Potential $v(r)$ in Example 4

PROBLEM SET 2.5

1–10 GENERAL SOLUTION

Find a real general solution, showing the details of your work.

1. $x^2y'' - 6y = 0$
2. $4x^2y'' + 4xy' - y = 0$
3. $x^2y'' - 7xy' + 16y = 0$
4. $x^2y'' + 3xy' + y = 0$
5. $x^2y'' - xy' + 2y = 0$
6. $2x^2y'' + 4xy' + 5y = 0$
7. $(10x^2D^2 - 20xD + 22.4I)y = 0$
8. $(4x^2D^2 + I)y = 0$
9. $(100x^2D^2 + 9I)y = 0$
10. $(10x^2D^2 + 6xD + 0.5I)y = 0$

11–15 INITIAL VALUE PROBLEM

Solve and graph the solution, showing the details of your work.

11. $x^2y'' - 4xy' + 6y = 0$, $y(1) = 1$, $y'(1) = 0$
12. $x^2y'' + 3xy' + y = 0$, $y(1) = 4$, $y'(1) = -2$
13. $(x^2D^2 + 2xD + 100.25I)y = 0$, $y(1) = 2$, $y'(1) = -11$
14. $(x^2D^2 - 2xD + 2.25I)y = 0$, $y(1) = 2.2$, $y'(1) = 2.5$
15. $(xD^2 + 4D)y = 0$, $y(1) = 12$, $y'(1) = -6$

16. TEAM PROJECT. Double Root

(A) Derive a second linearly independent solution of (1) by reduction of order; but instead of using (9), Sec. 2.1, perform all steps directly for the present ODE (1).

(B) Obtain $x^m \ln x$ by considering the solutions x^m and x^{m+s} of a suitable Euler–Cauchy equation and letting $s \to 0$.

(C) Verify by substitution that $x^m \ln x$, $m = (1 - a)/2$, is a solution in the critical case.

(D) Transform the Euler–Cauchy equation (1) into an ODE with constant coefficients by setting $x = e^t$ $(x > 0)$.

(E) Obtain a second linearly independent solution of the Euler–Cauchy equation in the "critical case" from that of a constant-coefficient ODE.

2.6 Existence and Uniqueness of Solutions. Wronskian

In this section we shall discuss the general theory of homogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

with continuous, but otherwise arbitrary ***variable coefficients*** p and q. This will concern the existence and form of a general solution of (1) as well as the uniqueness of the solution of initial value problems consisting of such an ODE and two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \tag{2}$$

with given x_0, K_0, and K_1.

The two main results will be Theorem 1, stating that such an initial value problem always has a solution which is unique, and Theorem 4, stating that a general solution

$$y = c_1y_1 + c_2y_2 \qquad (c_1, c_2 \text{ arbitrary}) \tag{3}$$

includes all solutions. Hence *linear* ODEs with continuous coefficients have no *"singular solutions"* (solutions not obtainable from a general solution).

Clearly, no such theory was needed for constant-coefficient or Euler–Cauchy equations because everything resulted explicitly from our calculations.

Central to our present discussion is the following theorem.

THEOREM 1

Existence and Uniqueness Theorem for Initial Value Problems

If $p(x)$ and $q(x)$ are continuous functions on some open interval I (see Sec. 1.1) *and x_0 is in I, then the initial value problem consisting of* (1) *and* (2) *has a unique solution $y(x)$ on the interval I.*

The proof of existence uses the same prerequisites as the existence proof in Sec. 1.7 and will not be presented here; it can be found in Ref. [A11] listed in App. 1. Uniqueness proofs are usually simpler than existence proofs. But for Theorem 1, even the uniqueness proof is long, and we give it as an additional proof in App. 4.

Linear Independence of Solutions

Remember from Sec. 2.1 that a general solution on an open interval I is made up from a **basis** y_1, y_2 on I, that is, from a pair of linearly independent solutions on I. Here we call y_1, y_2 **linearly independent** on I if the equation

$$(4) \qquad k_1y_1(x) + k_2y_2(x) = 0 \quad \text{on } I \qquad \text{implies} \qquad k_1 = 0, \quad k_2 = 0.$$

We call y_1, y_2 **linearly dependent** on I if this equation also holds for constants k_1, k_2 not both 0. In this case, and only in this case, y_1 and y_2 are proportional on I, that is (see Sec. 2.1),

$$(5) \qquad \text{(a)} \quad y_1 = ky_2 \qquad \text{or} \qquad \text{(b)} \quad y_2 = ly_1 \qquad \text{for all } x \text{ on } I.$$

For our discussion the following criterion of linear independence and dependence of solutions will be helpful.

THEOREM 2

Linear Dependence and Independence of Solutions

Let the ODE (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval* I. *Then two solutions* y_1 *and* y_2 *of* (1) *on* I *are linearly dependent on* I *if and only if their* **"Wronskian"**

$$\textbf{(6)} \qquad W(y_1, y_2) = y_1y_2' - y_2y_1'$$

is 0 *at some* x_0 *in* I. *Furthermore, if* $W = 0$ *at an* $x = x_0$ *in* I, *then* $W \equiv 0$ *on* I; *hence if there is an* x_1 *in* I *at which* W *is not* 0, *then* y_1, y_2 *are linearly independent on* I.

PROOF **(a)** Let y_1 and y_2 be linearly dependent on I. Then (5a) or (5b) holds on I. If (5a) holds, then

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = ky_2y_2' - y_2ky_2' = 0.$$

Similarly if (5b) holds.

(b) Conversely, we let $W(y_1, y_2) = 0$ for some $x = x_0$ and show that this implies linear dependence of y_1 and y_2 on I. We consider the linear system of equations in the unknowns k_1, k_2

$$(7) \qquad \begin{aligned} k_1y_1(x_0) + k_2y_2(x_0) &= 0 \\ k_1y_1'(x_0) + k_2y_2'(x_0) &= 0. \end{aligned}$$

To eliminate k_2, multiply the first equation by y_2' and the second by $-y_2$ and add the resulting equations. This gives

$$k_1y_1(x_0)y_2'(x_0) - k_1y_1'(x_0)y_2(x_0) = k_1W(y_1(x_0), y_2(x_0)) = 0.$$

Similarly, to eliminate k_1, multiply the first equation by $-y_1'$ and the second by y_1 and add the resulting equations. This gives

$$k_2W(y_1(x_0), y_2(x_0)) = 0.$$

If W were not 0 at x_0, we could divide by W and conclude that $k_1 = k_2 = 0$. Since W is 0, division is not possible, and the system has a solution for which k_1 and k_2 are not both 0. Using *these numbers* k_1, k_2, we introduce the function

$$y(x) = k_1 y_1(x) + k_2 y_2(x).$$

Since (1) is homogeneous linear, Fundamental Theorem 1 in Sec. 2.1 (the superposition principle) implies that this function is a solution of (1) on I. From (7) we see that it satisfies the initial conditions $y(x_0) = 0$, $y'(x_0) = 0$. Now another solution of (1) satisfying the same initial conditions is $y^* \equiv 0$. Since the coefficients p and q of (1) are continuous, Theorem 1 applies and gives uniqueness, that is, $y \equiv y^*$, written out

$$k_1 y_1 + k_2 y_2 \equiv 0 \qquad \text{on } I.$$

Now since k_1 and k_2 are not both zero, this means linear dependence of y_1, y_2 on I.

(c) We prove the last statement of the theorem. If $W(x_0) = 0$ at an x_0 in I, we have linear dependence of y_1, y_2 on I by part (b), hence $W \equiv 0$ by part (a) of this proof. Hence in the case of linear dependence it cannot happen that $W(x_1) \neq 0$ at an x_1 in I. If it does happen, it thus implies linear independence as claimed. ■

Remark. Determinants. Students familiar with second-order determinants may have noticed that

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

This determinant is called the *Wronski determinant*[5] or, briefly, the **Wronskian,** of two solutions y_1 and y_2 of (1), as has already been mentioned in (6). Note that its four entries occupy the same positions as in the linear system (7).

EXAMPLE 1 Illustration of Theorem 2

The functions $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of $y'' + \omega^2 y = 0$. Their Wronskian is

$$W(\cos \omega x, \sin \omega x) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = y_1 y_2' - y_2 y_1' = \omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega.$$

Theorem 2 shows that these solutions are linearly independent if and only if $\omega \neq 0$. Of course, we can see this directly from the quotient $y_2/y_1 = \tan \omega x$. For $\omega = 0$ we have $y_2 \equiv 0$, which implies linear dependence (why?). ■

EXAMPLE 2 Illustration of Theorem 2 for a Double Root

A general solution of $y'' - 2y' + y = 0$ on any interval is $y = (c_1 + c_2 x)e^x$. (Verify!). The corresponding Wronskian is not 0, which shows linear independence of e^x and xe^x on any interval. Namely,

$$W(x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0.$$ ■

[5]Introduced by WRONSKI (JOSEF MARIA HÖNE, 1776–1853), Polish mathematician.

A General Solution of (1) Includes All Solutions

This will be our second main result, as announced at the beginning. Let us start with existence.

THEOREM 3

Existence of a General Solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then (1) *has a general solution on I.*

PROOF By Theorem 1, the ODE (1) has a solution $y_1(x)$ on I satisfying the initial conditions

$$y_1(x_0) = 1, \qquad y_1'(x_0) = 0$$

and a solution $y_2(x)$ on I satisfying the initial conditions

$$y_2(x_0) = 0, \qquad y_2'(x_0) = 1.$$

The Wronskian of these two solutions has at $x = x_0$ the value

$$W(y_1(0), y_2(0)) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 1.$$

Hence, by Theorem 2, these solutions are linearly independent on I. They form a basis of solutions of (1) on I, and $y = c_1y_1 + c_2y_2$ with arbitrary c_1, c_2 is a general solution of (1) on I, whose existence we wanted to prove. ■

We finally show that a general solution is as general as it can possibly be.

THEOREM 4

A General Solution Includes All Solutions

If the ODE (1) *has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution $y = Y(x)$ of* (1) *on I is of the form*

$$Y(x) = C_1y_1(x) + C_2y_2(x) \tag{8}$$

where y_1, y_2 is any basis of solutions of (1) *on I and C_1, C_2 are suitable constants.*

Hence (1) *does not have* **singular solutions** *(that is, solutions not obtainable from a general solution).*

PROOF Let $y = Y(x)$ be any solution of (1) on I. Now, by Theorem 3 the ODE (1) has a general solution

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{9}$$

on I. We have to find suitable values of c_1, c_2 such that $y(x) = Y(x)$ on I. We choose any x_0 in I and show first that we can find values of c_1, c_2 such that we reach agreement at x_0, that is, $y(x_0) = Y(x_0)$ and $y'(x_0) = Y'(x_0)$. Written out in terms of (9), this becomes

$$(10) \qquad \begin{aligned} &\text{(a)} \quad c_1y_1(x_0) + c_2y_2(x_0) = Y(x_0) \\ &\text{(b)} \quad c_1y_1'(x_0) + c_2y_2'(x_0) = Y'(x_0). \end{aligned}$$

We determine the unknowns c_1 and c_2. To eliminate c_2, we multiply (10a) by $y_2'(x_0)$ and (10b) by $-y_2(x_0)$ and add the resulting equations. This gives an equation for c_1. Then we multiply (10a) by $-y_1'(x_0)$ and (10b) by $y_1(x_0)$ and add the resulting equations. This gives an equation for c_2. These new equations are as follows, where we take the values of y_1, y_1', y_2, y_2', Y, Y' at x_0,

$$c_1(y_1y_2' - y_2y_1') = c_1W(y_1, y_2) = Yy_2' - y_2Y'$$
$$c_2(y_1y_2' - y_2y_1') = c_2W(y_1, y_2) = y_1Y' - Yy_1'.$$

Since y_1, y_2 is a basis, the Wronskian W in these equations is not 0, and we can solve for c_1 and c_2. We call the (unique) solution $c_1 = C_1$, $c_2 = C_2$. By substituting it into (9) we obtain from (9) the particular solution

$$y^*(x) = C_1y_1(x) + C_2y_2(x).$$

Now since C_1, C_2 is a solution of (10), we see from (10) that

$$y^*(x_0) = Y(x_0), \qquad y^{*\prime}(x_0) = Y'(x_0).$$

From the uniqueness stated in Theorem 1 this implies that y^* and Y must be equal everywhere on I, and the proof is complete. ■

Looking back at he content of this section, we see that homogeneous linear ODEs with continuous variable coefficients have a conceptually and structurally rather transparent existence and uniqueness theory of solutions. Important in itself, this theory will also provide the foundation of an investigation of nonhomogeneous linear ODEs, whose theory and engineering applications we shall study in the remaining four sections of this chapter.

PROBLEM SET 2.6

1–17 BASES OF SOLUTIONS. CORRESPONDING ODEs. WRONSKIANS

Find an ODE (1) for which the given functions are solutions. Show linear independence (a) by considering quotients, (b) by Theorem 2.

1. $e^{0.5x}$, $e^{-0.5x}$

2. $\cos \pi x$, $\sin \pi x$

3. e^{kx}, xe^{kx}

4. x^3, x^{-2}

5. $x^{0.25}$, $x^{0.25} \ln x$

6. $e^{3.4x}$, $e^{-2.5x}$

7. $\cos(2 \ln x)$, $\sin(2 \ln x)$

8. e^{-2x}, xe^{-2x}

9. $x^{1.5}$, $x^{-0.5}$

10. x^{-3}, $x^{-3} \ln x$

11. $\cosh 2.5x$, $\sinh 2.5x$

12. $e^{-2x} \cos \omega x$, $e^{-2x} \sin \omega x$

13. $e^{-x} \cos 0.8x$, $e^{-x} \sin 0.8x$

14. $x^{-1} \cos(\ln x)$, $x^{-1} \sin(\ln x)$

15. $e^{-2.5x} \cos 0.3x$, $e^{-2.5x} \sin 0.3x$

16. $e^{-kx} \cos \pi x$, $e^{-kx} \sin \pi x$

17. $e^{-3.8\pi x}$, $xe^{-3.8\pi x}$

18. TEAM PROJECT. Consequences of the Present Theory. This concerns some noteworthy general properties of solutions. Assume that the coefficients p and q of the ODE (1) are continuous on some open interval I, to which the subsequent statements refer.

(A) Solve $y'' - y = 0$ (a) by exponential functions, (b) by hyperbolic functions. How are the constants in the corresponding general solutions related?

(B) Prove that the solutions of a basis cannot be 0 at the same point.

(C) Prove that the solutions of a basis cannot have a maximum or minimum at the same point.

(D) Express $(y_2/y_1)'$ by a formula involving the Wronskian W. Why is it likely that such a formula should exist? Use it to find W in Prob. 10.

(E) Sketch $y_1(x) = x^3$ if $x \geqq 0$ and 0 if $x < 0$, $y_2(x) = 0$ if $x \geqq 0$ and x^3 if $x < 0$. Show linear independence on $-1 < x < 1$. What is their Wronskian? What Euler–Cauchy equation do y_1, y_2 satisfy? Is there a contradiction to Theorem 2?

(F) Prove **Abel's formula**[6]

$$W(y_1(x), y_2(x)) = c \exp\left[-\int_{x_0}^{x} p(t)\, dt\right]$$

where $c = W(y_1(x_0), y_2(x_0))$. Apply it to Prob. 12. *Hint:* Write (1) for y_1 and for y_2. Eliminate q algebraically from these two ODEs, obtaining a first-order linear ODE. Solve it.

2.7 Nonhomogeneous ODEs

Method of Undetermined Coefficients

In this section we proceed from homogeneous to nonhomogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

where $r(x) \not\equiv 0$. We shall see that a "general solution" of (1) is the sum of a general solution of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and a "particular solution" of (1). These two new terms "general solution of (1)" and "particular solution of (1)" are defined as follows.

DEFINITION

General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x); \tag{3}$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Our task is now twofold, first to justify these definitions and then to develop a method for finding a solution y_p of (1).

Accordingly, we first show that a general solution as just defined satisfies (1) and that the solutions of (1) and (2) are related in a very simple way.

THEOREM 1

Relations of Solutions of (1) to Those of (2)

(a) *The sum of a solution y of* (1) *on some open interval I and a solution $\tilde{y}$ of* (2) *on I is a solution of* (1) *on I. In particular,* (3) *is a solution of* (1) *on I.*

(b) *The difference of two solutions of* (1) *on I is a solution of* (2) *on I.*

[6] NIELS HENRIK ABEL (1802–1829), Norwegian mathematician.

PROOF (a) Let $L[y]$ denote the left side of (1). Then for any solutions y of (1) and $\tilde{y}$ of (2) on I,

$$L[y + \tilde{y}] = L[y] + L[\tilde{y}] = r + 0 = r.$$

(b) For any solutions y and y^* of (1) on I we have $L[y - y^*] = L[y] - L[y^*] = r - r = 0$. ■

Now for *homogeneous ODEs* (2) we know that general solutions include all solutions. We show that the same is true for nonhomogeneous ODEs (1).

THEOREM 2

A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) *are continuous on some open interval I, then every solution of* (1) *on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution* (3) *of* (1) *on I.*

PROOF Let y^* be any solution of (1) on I and x_0 any x in I. Let (3) be any general solution of (1) on I. This solution exists. Indeed, $y_h = c_1y_1 + c_2y_2$ exists by Theorem 3 in Sec. 2.6 because of the continuity assumption, and y_p exists according to a construction to be shown in Sec. 2.10. Now, by Theorem 1(b) just proved, the difference $Y = y^* - y_p$ is a solution of (2) on I. At x_0 we have

$$Y(x_0) = y^*(x_0) - y_p(x_0), \qquad Y'(x_0) = y^{*\prime}(x_0) - y_p'(x_0).$$

Theorem 1 in Sec. 2.6 implies that for these conditions, as for any other initial conditions in I, there exists a unique particular solution of (2) obtained by assigning suitable values to c_1, c_2 in y_h. From this and $y^* = Y + y_p$ the statement follows. ■

Method of Undetermined Coefficients

Our discussion suggests the following. *To solve the nonhomogeneous ODE* (1) *or an initial value problem for* (1), *we have to solve the homogeneous ODE* (2) *and find any solution* y_p *of* (1), so that we obtain a general solution (3) of (1).

How can we find a solution y_p of (1)? One method is the so-called **method of undetermined coefficients.** It is much simpler than another, more general method (to be discussed in Sec. 2.10). Since it applies to models of vibrational systems and electric circuits to be shown in the next two sections, it is frequently used in engineering.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with ***constant coefficients a and b***

$$y'' + ay' + by = r(x) \tag{4}$$

when $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea. We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. Table 2.1 on p. 80 shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Choice Rules for the Method of Undetermined Coefficients

(a) **Basic Rule.** *If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (4).*

(b) **Modification Rule.** *If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply your choice of y_p by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).*

(c) **Sum Rule.** *If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.*

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$. (Verify!)

The method is self-correcting. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Let us illustrate Rules (a)–(c) by the typical Examples 1–3.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

EXAMPLE 1 Application of the Basic Rule (a)

Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5. \tag{5}$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The ODE $y'' + y = 0$ has the general solution

$$y_h = A \cos x + B \sin x.$$

Step 2. Solution y_p of the nonhomogeneous ODE. We first try $y_p = Kx^2$. Then $y_p'' = 2K$. By substitution, $2K + Kx^2 = 0.001x^2$. For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides; thus $K = 0.001$ and $2K = 0$, a contradiction.

The second line in Table 2.1 suggests the choice

$$y_p = K_2 x^2 + K_1 x + K_0. \qquad \text{Then} \qquad y_p'' + y_p = 2K_2 + K_2 x^2 + K_1 x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have $K_2 = 0.001$, $K_1 = 0$, $2K_2 + K_0 = 0$. Hence $K_0 = -2K_2 = -0.002$. This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A \cos x + B \sin x + 0.001x^2 - 0.002.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \qquad \text{and} \qquad y'(0) = B = 1.5.$$

This gives the answer (Fig. 49)

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$$

Figure 49 shows y as well as the quadratic parabola y_p about which y is oscillating, practically like a sine curve since the cosine term is smaller by a factor of about 1/1000. ■

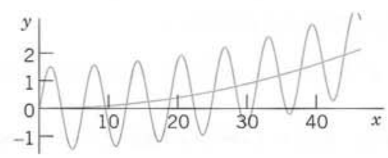


Fig. 49. Solution in Example 1

EXAMPLE 2 Application of the Modification Rule (b)

Solve the initial value problem

$$(6) \qquad y'' + 3y' + 2.25y = -10\, e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}.$$

Step 2. Solution y_p of the nonhomogeneous ODE. The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a *double root* of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2 e^{-1.5x}. \qquad \text{Then} \qquad y_p' = C(2x - 1.5x^2)e^{-1.5x}, \qquad y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}.$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^2 e^{-1.5x}$. Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2 x)e^{-1.5x} - 5x^2 e^{-1.5x}.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives

$$y' = (c_2 - 1.5c_1 - 1.5c_2 x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2 e^{-1.5x}.$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer (Fig. 50)

$$y = (1 + 1.5x)\, e^{-1.5x} - 5x^2 e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases. ■

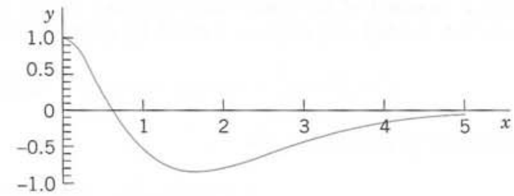


Fig. 50. Solution in Example 2

EXAMPLE 3 Application of the Sum Rule (c)

Solve the initial value problem

$$y'' + 2y' + 5y = e^{0.5x} + 40 \cos 10x - 190 \sin 10x, \qquad y(0) = 0.16, \qquad y'(0) = 40.08. \tag{7}$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation

$$\lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i) = 0$$

shows that a real general solution of the homogeneous ODE is

$$y_h = e^{-x}(A \cos 2x + B \sin 2x).$$

Step 2. Solution of the nonhomogeneous ODE. We write $y_p = y_{p1} + y_{p2}$, where y_{p1} corresponds to the exponential term and y_{p2} to the sum of the other two terms. We set

$$y_{p1} = Ce^{0.5x}. \qquad \text{Then} \qquad y'_{p1} = 0.5Ce^{0.5x} \qquad \text{and} \qquad y''_{p1} = 0.25Ce^{0.5x}.$$

Substitution into the given ODE and omission of the exponential factor gives $(0.25 + 2 \cdot 0.5 + 5)C = 1$, hence $C = 1/6.25 = 0.16$, and $y_{p1} = 0.16e^{0.5x}$.

We now set $y_{p2} = K \cos 10x + M \sin 10x$, as in Table 2.1, and obtain

$$y'_{p2} = -10K \sin 10x + 10M \cos 10x, \qquad y''_{p2} = -100K \cos 10x - 100M \sin 10x.$$

Substitution into the given ODE gives for the cosine terms and for the sine terms

$$-100K + 2 \cdot 10M + 5K = 40, \qquad -100M - 2 \cdot 10K + 5M = -190$$

or, by simplification,

$$-95K + 20M = 40, \qquad -20K - 95M = -190.$$

The solution is $K = 0$, $M = 2$. Hence $y_{p2} = 2 \sin 10x$. Together,

$$y = y_h + y_{p1} + y_{p2} = e^{-x}(A \cos 2x + B \sin 2x) + 0.16e^{0.5x} + 2 \sin 10x.$$

Step 3. Solution of the initial value problem. From y and the first initial condition, $y(0) = A + 0.16 = 0.16$, hence $A = 0$. Differentiation gives

$$y' = e^{-x}(-A \cos 2x - B \sin 2x - 2A \sin 2x + 2B \cos 2x) + 0.08e^{0.5x} + 20 \cos 10x.$$

From this and the second initial condition we have $y'(0) = -A + 2B + 0.08 + 20 = 40.08$, hence $B = 10$. This gives the solution (Fig. 51)

$$y = 10e^{-x} \sin 2x + 0.16e^{0.5x} + 2 \sin 10x.$$

The first term goes to 0 relatively fast. When $x = 4$, it is practically 0, as the dashed curves $\pm 10e^{-x} + 0.16e^{0.5x}$ show. From then on, the last term, $2 \sin 10x$, gives an oscillation about $0.16e^{0.5x}$, the monotone increasing dashed curve. ■

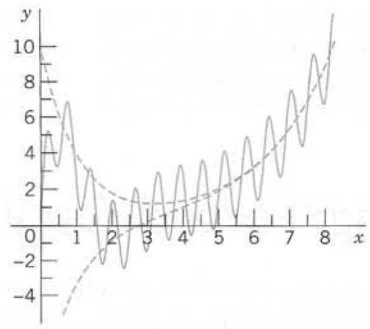


Fig. 51. Solution in Example 3

Stability. The following is important. If (and only if) all the roots of the characteristic equation of the homogeneous ODE $y'' + ay' + by = 0$ in (4) are negative, or have a negative real part, then a general solution y_h of this ODE goes to 0 as $x \to \infty$, so that the **"transient solution"** $y = y_h + y_p$ of (4) approaches the **"steady-state solution"** y_p. In this case the nonhomogeneous ODE and the physical or other system modeled by the ODE are called **stable;** otherwise they are called **unstable.** For instance, the ODE in Example 1 is unstable.

Basic applications follow in the next two sections.

PROBLEM SET 2.7

1–14 GENERAL SOLUTIONS OF NONHOMOGENEOUS ODEs

Find a (real) general solution. Which rule are you using? (Show each step of your calculation.)

1. $y'' + 3y' + 2y = 30e^{2x}$
2. $y'' + 4y' + 3.75y = 109 \cos 5x$
3. $y'' - 16y = 19.2e^{4x} + 60e^{x}$
4. $y'' + 9y = \cos x + \frac{1}{3} \cos 3x$
5. $y'' + y' - 6y = 6x^3 - 3x^2 + 12x$
6. $y'' + 4y' + 4y = e^{-2x} \sin 2x$
7. $y'' + 6y' + 73y = 80e^{x} \cos 4x$
8. $y'' + 10y' + 25y = 100 \sinh 5x$
9. $y'' - 0.16y = 32 \cosh 0.4x$
10. $y'' + 4y' + 6.25y = 3.125(x + 1)^2$
11. $y'' + 1.44y = 24 \cos 1.2x$
12. $y'' + 9y = 18x + 36 \sin 3x$
13. $y'' + 4y' + 5y = 25x^2 + 13 \sin 2x$
14. $y'' + 2y' + y = 2x \sin x$

15–20 INITIAL VALUE PROBLEMS FOR NONHOMOGENEOUS ODEs

Solve the initial value problem. State which rules you are using. Show each step of your calculation in detail.

15. $y'' + 4y = 16 \cos 2x, \quad y(0) = 0, \quad y'(0) = 0$
16. $y'' - 3y' + 2.25y = 27(x^2 - x)$, $y(0) = 20, \quad y'(0) = 30$
17. $y'' + 0.2y' + 0.26y = 1.22e^{0.5x}$, $y(0) = 3.5, \quad y'(0) = 0.35$
18. $y'' - 2y' = 12e^{2x} - 8e^{-2x}$, $y(0) = -2, \quad y'(0) = 12$
19. $y'' - y' - 12y = 144x^3 + 12.5$, $y(0) = 5, \quad y'(0) = -0.5$
20. $y'' + 2y' + 10y = 17 \sin x - 37 \sin 3x$, $y(0) = 6.6, \quad y'(0) = -2.2$
21. **WRITING PROJECT. Initial Value Problem.** Write out all the details of Example 3 in your own words. Discuss Fig. 51 in more detail. Why is it that some of the "half-waves" do not reach the dashed curves, whereas others preceding them (and, of course, all later ones) excede the dashed curves?
22. **TEAM PROJECT. Extensions of the Method of Undetermined Coefficients. (a)** Extend the method to products of the function in Table 2.1. **(b)** Extend the method to Euler–Cauchy equations. Comment on the practical significance of such extensions.
23. **CAS PROJECT. Structure of Solutions of Initial Value Problems.** Using the present method, find, graph, and discuss the solutions y of initial value problems of your own choice. Explore effects on solutions caused by

changes of initial conditions. Graph y_p, y, $y - y_p$ separately, to see the separate effects. Find a problem in which (a) the part of y resulting from y_h decreases to zero, (b) increases, (c) is not present in the answer y. Study a problem with $y(0) = 0$, $y'(0) = 0$. Consider a problem in which you need the Modification Rule (a) for a simple root, (b) for a double root. Make sure that your problems cover all three Cases I, II, III (see Sec. 2.2).

2.8 Modeling: Forced Oscillations. Resonance

In Sec. 2.4 we considered vertical motions of a mass-spring system (vibration of a mass m on an elastic spring, as in Figs. 32 and 52) and modeled it by the ***homogeneous*** linear ODE

$$my'' + cy' + ky = 0. \tag{1}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest. These were **free motions,** that is, motions in the absence of *external forces* (outside forces) caused solely by *internal forces,* forces within the system. These are the force of inertia my'', the damping force cy' (if $c > 0$), and the spring force ky acting as a restoring force.

We now extend our model by including an external force, call it $r(t)$, on the right. Then we have

$$my'' + cy' + ky = r(t). \tag{2*}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force,** and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0,\ \omega > 0).$$

Then we have the nonhomogeneous ODE

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2}$$

Its solution will familiarize us with further interesting facts fundamental in engineering mathematics, in particular with resonance.

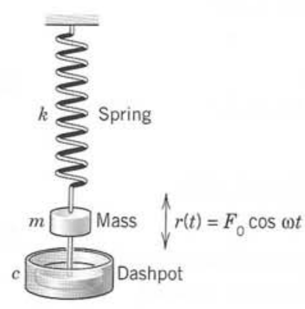


Fig. 52. Mass on a spring

Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution y_h of the homogeneous ODE (1) plus any solution y_p of (2). To find y_p, we use the method of undetermined coefficients (Sec. 2.7), starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{3}$$

By differentiating this function (chain rule!) we obtain

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$

$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_p, y_p', and y_p'' into (2) and collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cosine terms on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right. This gives the two equations

$$\begin{aligned} (k - m\omega^2)a + \quad \omega cb \quad &= F_0 \\ -\omega ca \quad + (k - m\omega^2)b &= 0 \end{aligned} \tag{4}$$

for determining the unknown coefficients a and b. This is a linear system. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply the first equation by ωc and the second by $k - m\omega^2$ and add to get

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(k - m\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$ (> 0) as in Sec. 2.4, then $k = m\omega_0^2$ and we obtain

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{5}$$

We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

$$y(t) = y_h(t) + y_p(t). \tag{6}$$

Here y_h is a general solution of the homogeneous ODE (1) and y_p is given by (3) with coefficients (5).

We shall now discuss the behavior of the mechanical system, distinguishing between the two cases $c = 0$ (no damping) and $c > 0$ (damping). These cases will correspond to two basically different types of output.

Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0^2 = k/m$)

$$(7) \qquad y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t.$$

Here we must assume that $\omega^2 \neq \omega_0^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4]. From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

$$\textbf{(8)} \qquad y(t) = C \cos (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

We see that this output is a ***superposition of two harmonic oscillations*** *of the frequencies just mentioned.*

Resonance. We discuss (7). We see that the maximum amplitude of y_p is (put $\cos \omega t = 1$)

$$(9) \qquad a_0 = \frac{F_0}{k} \rho \qquad \text{where} \qquad \rho = \frac{1}{1 - (\omega/\omega_0)^2} .$$

a_0 depends on ω and ω_0. If $\omega \to \omega_0$, then ρ and a_0 tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance.** ρ is called the **resonance factor** (Fig. 53), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution y_p and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

In the case of resonance the nonhomogeneous ODE (2) becomes

$$\textbf{(10)} \qquad y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t.$$

Then (7) is no longer valid, and from the Modification Rule in Sec. 2.7 we conclude that a particular solution of (10) is of the form

$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

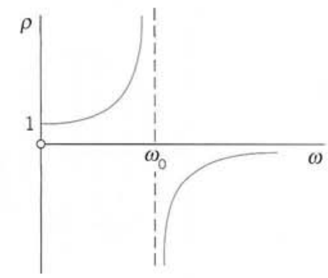


Fig. 53. Resonance factor $\rho(\omega)$

By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 54)

$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \tag{11}$$

We see that because of the factor t the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system. We shall return to this practical aspect of resonance later in this section.

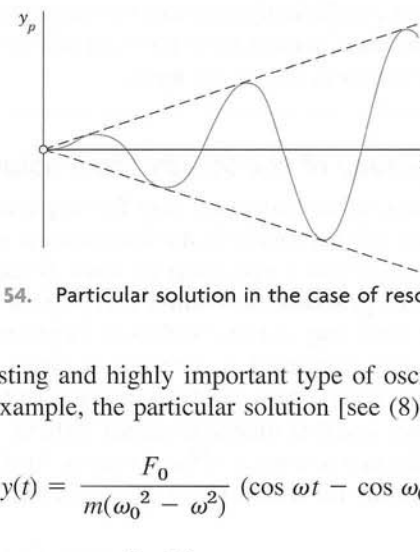


Fig. 54. Particular solution in the case of resonance

Beats. Another interesting and highly important type of oscillation is obtained if ω is close to ω_0. Take, for example, the particular solution [see (8)]

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \qquad (\omega \neq \omega_0). \tag{12}$$

Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

Since ω is close to ω_0, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 55, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

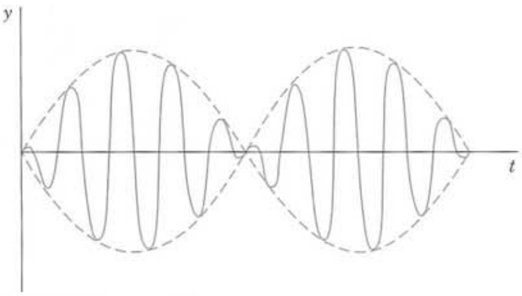


Fig. 55. Forced undamped oscillation when the difference of the input and natural frequencies is small **("beats")**

Case 2. Damped Forced Oscillations

If the damping of the mass-spring system is not negligibly small, we have $c > 0$ and a damping term cy' in (1) and (2). Then the general solution y_h of the homogeneous ODE (1) approaches zero as t goes to infinity, as we know from Sec. 2.4. Practically, it is zero after a sufficiently long time. Hence the **"transient solution"** (6) of (2), given by $y = y_h + y_p$, approaches the **"steady-state solution"** y_p. This proves the following.

THEOREM 1

Steady-State Solution

After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force [see (2)] *will practically be a harmonic oscillation whose frequency is that of the input.*

Amplitude of the Steady-State Solution. Practical Resonance

Whereas in the undamped case the amplitude of y_p approaches infinity as ω approaches ω_0, this will not happen in the damped case. In this case the amplitude will always be finite. But it may have a maximum for some ω depending on the damping constant c. This may be called **practical resonance.** It is of great importance because if c is not too large, then some input may excite oscillations large enough to damage or even destroy the system. Such cases happened, in particular in earlier times when less was known about resonance. Machines, cars, ships, airplanes, bridges, and high-rising buildings are vibrating mechanical systems, and it is sometimes rather difficult to find constructions that are completely free of undesired resonance effects, caused, for instance, by an engine or by strong winds.

To study the amplitude of y_p as a function of ω, we write (3) in the form

$$y_p(t) = C^* \cos(\omega t - \eta). \tag{13}$$

C^* is called the **amplitude** of y_p and η the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

(14)
$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0{}^2 - \omega^2)^2 + \omega^2 c^2}},$$
$$\tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0{}^2 - \omega^2)}.$$

Let us see whether $C^*(\omega)$ has a maximum and, if so, find its location and then its size. We denote the radicand in the second root in C^* by R. Equating the derivative of C^* to zero, we obtain

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2} R^{-3/2}\right)[2m^2({\omega_0}^2 - \omega^2)(-2\omega) + 2\omega c^2].$$

The expression in the brackets [. . .] is zero if

$$c^2 = 2m^2({\omega_0}^2 - \omega^2) \qquad ({\omega_0}^2 = k/m). \tag{15}$$

By reshuffling terms we have

$$2m^2\omega^2 = 2m^2{\omega_0}^2 - c^2 = 2mk - c^2.$$

The right side of this equation becomes negative if $c^2 > 2mk$, so that then (15) has no real solution and C^* decreases monotone as ω increases, as the lowest curve in Fig. 56 on p. 90 shows. If c is smaller, $c^2 < 2mk$, then (15) has a real solution $\omega = \omega_{max}$, where

$$\omega_{max}^2 = {\omega_0}^2 - \frac{c^2}{2m^2}. \tag{15*}$$

From (15*) we see that this solution increases as c decreases and approaches ω_0 as c approaches zero. See also Fig. 56.

The size of $C^*(\omega_{max})$ is obtained from (14), with $\omega^2 = \omega_{max}^2$ given by (15*). For this ω^2 we obtain in the second radicand in (14) from (15*)

$$m^2({\omega_0}^2 - \omega_{max}^2)^2 = \frac{c^4}{4m^2} \qquad \text{and} \qquad \omega_{max}^2 c^2 = \left({\omega_0}^2 - \frac{c^2}{2m^2}\right)c^2.$$

The sum of the right sides of these two formulas is

$$(c^4 + 4m^2{\omega_0}^2c^2 - 2c^4)/(4m^2) = c^2(4m^2{\omega_0}^2 - c^2)/(4m^2).$$

Substitution into (14) gives

$$C^*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4m^2{\omega_0}^2 - c^2}}. \tag{16}$$

We see that $C^*(\omega_{max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2{\omega_0}^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as c^2 ($< 2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1. Figure 56 shows the **amplification** C^*/F_0 (ratio of the amplitudes of output and input) as a function of ω for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant c.

Figure 57 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.

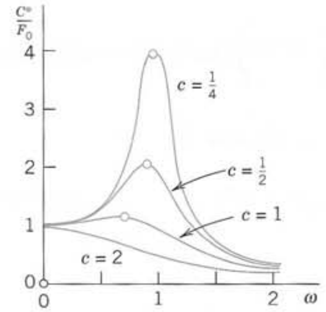


Fig. 56. Amplification C^*/F_0 as a function of ω for $m = 1$, $k = 1$, and various values of the damping constant c

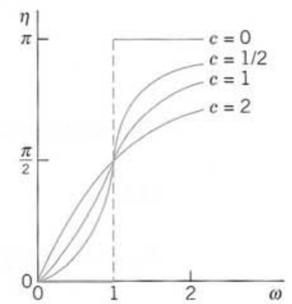


Fig. 57. Phase lag η as a function of ω for $m = 1$, $k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

PROBLEM SET 2.8

1–8 STEADY-STATE SOLUTIONS

Find the steady-state oscillation of the mass–spring system modeled by the given ODE. Show the details of your calculations.

1. $y'' + 6y' + 8y = 130 \cos 3t$
2. $4y'' + 8y' + 13y = 8 \sin 1.5t$
3. $y'' + y' + 4.25y = 221 \cos 4.5t$
4. $y'' + 4y' + 5y = \cos t - \sin t$
5. $(D^2 + 2D + I)y = -\sin 2t$
6. $(D^2 + 4D + 3I)y = \cos t + \frac{1}{3} \cos 3t$
7. $(D^2 + 6D + 18I)y = \cos 3t - 3 \sin 3t$
8. $(D^2 + 2D + 10I)y = -25 \sin 4t$

9–14 TRANSIENT SOLUTIONS

Find the transient motion of the mass–spring system modeled by the given ODE. (Show the details of your work.)

9. $y'' + 2y' + 0.75y = 13 \sin t$
10. $y'' + 4y' + 4y = \cos 4t$
11. $4y'' + 12y' + 9y = 75 \sin 3t$
12. $(D^2 + 5D + 4I)y = \sin 2t$
13. $(D^2 + 3D + 3.25I)y = 13 - 39 \cos 2t$
14. $(D^2 + 2D + 5I)y = 1 + \sin t$

15–20 INITIAL VALUE PROBLEMS

Find the motion of the mass–spring system modeled by the ODE and initial conditions. Sketch or graph the solution curve. In addition, sketch or graph the curve of $y - y_p$ to see when the system practically reaches the steady state.

15. $y'' + 2y' + 26y = 13 \cos 3t$, $y(0) = 1$, $y'(0) = 0.4$
16. $y'' + 64y = \cos t$, $y(0) = 0$, $y'(0) = 1$
17. $y'' + 6y' + 8y = 4 \sin 2t$, $y(0) = 0.7$, $y'(0) = -11.8$
18. $(D^2 + 2D + I)y = 75(\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t)$, $y(0) = 0$, $y'(0) = 1$
19. $(4D^2 + 12D + 13I)y = 12 \cos t - 6 \sin t$, $y(0) = 1$, $y'(0) = -1$
20. $y'' + 25y = 99 \cos 4.9t$, $y(0) = 2$, $y'(0) = 0$
21. **(Beats)** Derive the formula after (12) from (12). Can there be beats if the system has damping?
22. **(Beats)** How does the graph of the solution in Prob. 20 change if you change (a) $y(0)$, (b) the frequency of the driving force?
23. **WRITING PROJECT. Free and Forced Vibrations.** Write a condensed report of 2–3 pages on the most important facts about free and forced vibrations.
24. **CAS EXPERIMENT. Undamped Vibrations.** **(a)** Solve the initial value problem $y'' + y = \cos \omega t$, $\omega^2 \neq 1$, $y(0) = 0$, $y'(0) = 0$. Show that the solution can be written

$$y(t) = \frac{2}{1 - \omega^2} \sin \left[\frac{1}{2}(1 + \omega)t\right] \times \sin \left[\frac{1}{2}(1 - \omega)t\right]. \tag{17}$$

(b) Experiment with (17) by changing ω to see the change of the curves from those for small ω (> 0) to beats, to resonance and to large values of ω (see Fig. 58).

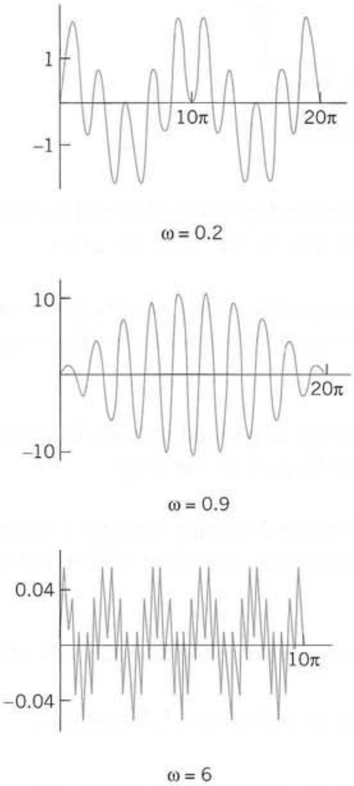


Fig. 58. Typical solution curves in CAS Experiment 24

25. TEAM PROJECT. Practical Resonance. (a) Give a detailed derivation of the crucial formula (16).

(b) By considering dC^*/dc show that $C^*(\omega_{\max})$ increases as c ($\leqq \sqrt{2mk}$) decreases.

(c) Illustrate practical resonance with an ODE of your own in which you vary c, and sketch or graph corresponding curves as in Fig. 56.

(d) Take your ODE with c fixed and an input of two terms, one with frequency close to the practical resonance frequency and the other not. Discuss and sketch or graph the output.

(e) Give other applications (not in the book) in which resonance is important.

26. (Gun barrel) Solve

$$y'' + y = \begin{cases} 1 - t^2/\pi^2 & \text{if } 0 \leqq t \leqq \pi \\ 0 & \text{if } t > \pi, \end{cases}$$

$$y(0) = y'(0) = 0.$$

This models an undamped system on which a force F acts during some interval of time (see Fig. 59), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). *Hint.* At π both y and y' must be continuous.

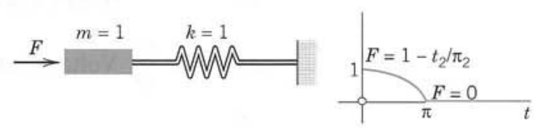


Fig. 59. Problem 26

2.9 Modeling: Electric Circuits

Designing good models is a task the computer cannot do. Hence setting up models has become an important task in modern applied mathematics. The best way to gain experience is to consider models from various fields. Accordingly, modeling electric circuits to be discussed will be *profitable for all students,* not just for electrical engineers and computer scientists.

We have just seen that linear ODEs have important applications in mechanics (see also Sec. 2.4). Similarly, they are models of electric circuits, as they occur as portions of large networks in computers and elsewhere. The circuits we shall consider here are basic building blocks of such networks. They contain three kinds of components, namely, resistors, inductors, and capacitors. Figure 60 on p. 92 shows such an ***RLC*-circuit,** as they are called. In it a resistor of resistance R Ω (ohms), an inductor of inductance L H (henrys), and a capacitor of capacitance C F (farads) are wired in series as shown, and connected to an electromotive force $E(t)$ V (volts) (a generator, for instance), sinusoidal as in Fig. 60, or of some other kind. R, L, C, and E are given and we want to find the current $I(t)$ A (amperes) in the circuit.

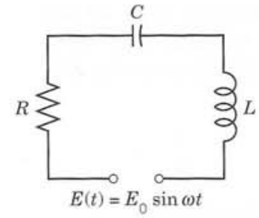


Fig. 60. *RLC*-circuit

An ODE for the current $I(t)$ in the *RLC*-circuit in Fig. 60 is obtained from the following law (which is the analog of Newton's second law, as we shall see later).

Kirchhoff's Voltage Law (KVL).[7] *The voltage (the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.*

In Fig. 60 the circuit is a closed loop, and the impressed voltage $E(t)$ equals the sum of the voltage drops across the three elements R, L, C of the loop.

Voltage Drops. Experiments show that a current I flowing through a resistor, inductor or capacitor causes a voltage drop (voltage difference, measured in volts) at the two ends; these drops are

RI	**(Ohm's law)** Voltage drop for a resistor of resistance R ohms (Ω),
$LI' = L \dfrac{dI}{dt}$	Voltage drop for an inductor of inductance L henrys (H),
$\dfrac{Q}{C}$	Voltage drop for a capacitor of capacitance C farads (F).

Here Q coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt}, \qquad \text{equivalently,} \qquad Q(t) = \int I(t)\, dt.$$

This is summarized in Fig. 61.

According to KVL we thus have in Fig. 60 for an *RLC*-circuit with electromotive force $E(t) = E_0 \sin \omega t$ (E_0 constant) as a model the **"integro-differential equation"**

$$LI' + RI + \frac{1}{C}\int I\, dt = E(t) = E_0 \sin \omega t. \tag{1'}$$

---

[7]GUSTAV ROBERT KIRCHHOFF (1824–1887), German physicist. Later we shall also need **Kirchhoff's current law (KCL):**

At any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.

The units of measurement of electrical quantities are named after ANDRÉ MARIE AMPÈRE (1775–1836), French physicist, CHARLES AUGUSTIN DE COULOMB (1736–1806), French physicist and engineer, MICHAEL FARADAY (1791–1867), English physicist, JOSEPH HENRY (1797–1878), American physicist, GEORG SIMON OHM (1789–1854), German physicist, and ALESSANDRO VOLTA (1745–1827), Italian physicist.

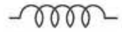
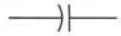
Name	Symbol	Notation	Unit	Voltage Drop
Ohm's resistor		R Ohm's resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L \dfrac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

Fig. 61. Elements in an *RLC*-circuit

To get rid of the integral, we differentiate (1′) with respect to t, obtaining

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_0\omega \cos \omega t. \tag{1}$$

This shows that the current in an *RLC*-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients.

From (1′), using $I = Q'$, hence $I' = Q''$, we also have directly

$$LQ'' + RQ' + \frac{1}{C} Q = E_0 \sin \omega t. \tag{1''}$$

But in most practical problems the current $I(t)$ is more important than the charge $Q(t)$, and for this reason we shall concentrate on (1) rather than on (1″).

Solving the ODE (1) for the Current. Discussion of Solution

A general solution of (1) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to (1) and I_p is a particular solution of (1). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$\begin{aligned} I_p &= a \cos \omega t + b \sin \omega t \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t) \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t) \end{aligned} \tag{2}$$

into (1). Then we collect the cosine terms and equate them to $E_0\omega \cos \omega t$ on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$L\omega^2(-a) + R\omega b + a/C = E_0\omega \qquad \text{(Cosine terms)}$$

$$L\omega^2(-b) + R\omega(-a) + b/C = 0 \qquad \text{(Sine terms).}$$

To solve this system for a and b, we first introduce a combination of L and C, called the **reactance**

$$S = \omega L - \frac{1}{\omega C}\,. \tag{3}$$

Dividing the previous two equations by ω, ordering them, and substituting S gives

$$-Sa + Rb = E_0$$
$$-Ra - Sb = 0.$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding. Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

In any practical case the resistance R is different from zero, so that we can solve for a and b,

$$\text{(4)} \qquad a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}.$$

Equation (2) with coefficients a and b given by (4) is the desired particular solution I_p of the nonhomogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal electromotive force.

Using (4), we can write I_p in terms of "physically visible" quantities, namely, amplitude I_0 and phase lag θ of the current behind the electromotive force, that is,

$$\textbf{(5)} \qquad I_p(t) = I_0 \sin(\omega t - \theta)$$

where [see (14) in App. A3.1]

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $\sqrt{R^2 + S^2}$ is called the **impedance.** Our formula shows that the impedance equals the ratio E_0/I_0. This is somewhat analogous to $E/I = R$ (Ohm's law).

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, where

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time. (This is as for the motion in the previous section.) Hence the transient current $I = I_h + I_p$ tends to the steady-state current I_p, and after some time the output will practically be a harmonic oscillation, which is given by (5) and whose frequency is that of the input (of the electromotive force).

EXAMPLE 1 ***RLC*-Circuit**

Find the current $I(t)$ in an *RLC*-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$F (farad), which is connected to a source of voltage $E(t) = 100 \sin 400t$ (hence $63\frac{2}{3}$ Hz $= 63\frac{2}{3}$ cycles/sec, because $400 = 63\frac{2}{3} \cdot 2\pi$). Assume that current and charge are zero when $t = 0$.

Solution. ***Step 1. General solution of the homogeneous ODE.*** Substituting R, L, C, and the derivative $E'(t)$ into (1), we obtain

$$0.1I'' + 11I' + 100I = 100 \cdot 400 \cos 400t.$$

Hence the homogeneous ODE is $0.1I'' + 11I' + 100I = 0$. Its characteristic equation is

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

Step 2. Particular solution I_p of (1). We calculate the reactance $S = 40 - 1/4 = 39.75$ and the steady-state current

$$I_p(t) = a \cos 400t + b \sin 400t$$

with coefficients obtained from (4)

$$a = \frac{-100 \cdot 39.75}{11^2 + 39.75^2} = -2.3368, \qquad b = \frac{100 \cdot 11}{11^2 + 39.75^2} = 0.6467.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t. \tag{6}$$

Step 3. Particular solution satisfying the initial conditions. How to use $Q(0) = 0$? We finally determine c_1 and c_2 from the initial conditions $I(0) = 0$ and $Q(0) = 0$. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.3368 = 0, \qquad \text{hence} \qquad c_2 = 2.3368 - c_1. \tag{7}$$

Furthermore, using (1′) with $t = 0$ and noting that the integral equals $Q(t)$ (see the formula before (1′)), we obtain

$$LI'(0) + R \cdot 0 + \frac{1}{C} \cdot 0 = 0, \qquad \text{hence} \qquad I'(0) = 0.$$

Differentiating (6) and setting $t = 0$, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.6467 \cdot 400 = 0, \qquad \text{hence} \qquad -10c_1 = 100(2.3368 - c_1) - 258.68.$$

The solution of this and (7) is $c_1 = -0.2776$, $c_2 = 2.6144$. Hence the answer is

$$I(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Figure 62 on p. 96 shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly. Thus after a very short time the current will practically execute harmonic oscillations of the input frequency $63\frac{2}{3}$ Hz $= 63\frac{2}{3}$ cycles/sec. Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.4246 \sin (400t - 1.3008).$$ ■

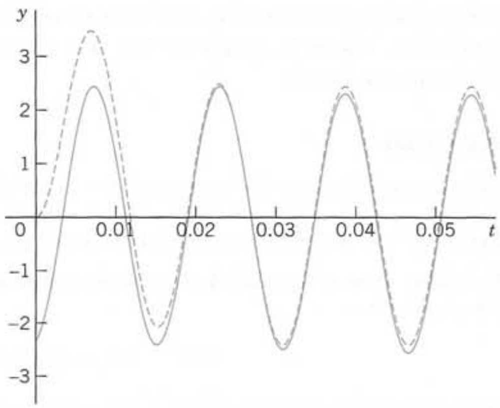


Fig. 62. Transient and steady-state currents in Example 1

Analogy of Electrical and Mechanical Quantities

Entirely different physical or other systems may have the same mathematical model. For instance, we have seen this from the various applications of the ODE $y' = ky$ in Chap. 1. Another impressive demonstration of this ***unifying power of mathematics*** is given by the ODE (1) for an electric *RLC*-circuit and the ODE (2) in the last section for a mass–spring system. Both equations

$$LI'' + RI' + \frac{1}{C} I = E_0\omega \cos \omega t \qquad \text{and} \qquad my'' + cy' + ky = F_0 \cos \omega t$$

are of the same form. Table 2.2 shows the analogy between the various quantities involved. The inductance L corresponds to the mass m and, indeed, an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance R corresponds to the damping constant c, and a resistor causes loss of energy, just as a damping dashpot does. And so on.

This analogy is ***strictly quantitative*** in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

The ***practical importance*** of this analogy is almost obvious. The analogy may be used for constructing an "electrical model" of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

Table 2.2 **Analogy of Electrical and Mechanical Quantities**

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$

PROBLEM SET 2.9

1. **(*RL*-circuit)** Model the *RL*-circuit in Fig. 63. Find the general solution when R, L, E are any constants. Graph or sketch solutions when $L = 0.1$ H, $R = 5\ \Omega$, $E = 12$ V.

2. **(*RL*-circuit)** Solve Prob. 1 when $E = E_0 \sin \omega t$ and R, L, E_0, ω are arbitrary. Sketch a typical solution.

3. **(*RC*-circuit)** Model the *RC*-circuit in Fig. 66. Find the current due to a constant E.

4. **(*RC*-circuit)** Find the current in the *RC*-circuit in Fig. 66 with $E = E_0 \sin \omega t$ and arbitrary R, C, E_0, and ω.

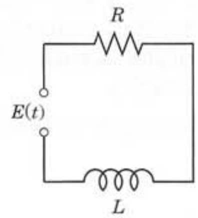


Fig. 63. *RL*-circuit

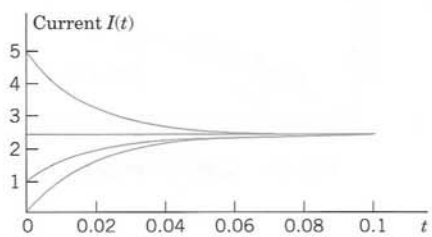


Fig. 64. Currents in Problem 1

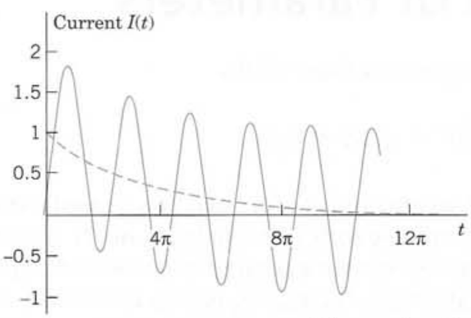


Fig. 65. Typical current $I = e^{-0.1t} + \sin(t - \frac{1}{4}\pi)$ in Problem 2

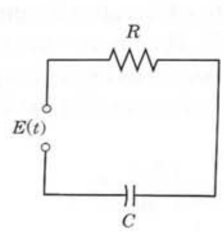


Fig. 66. *RC*-circuit

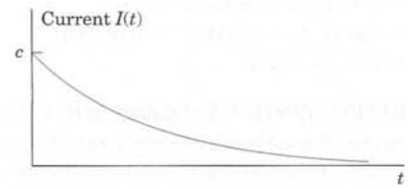


Fig. 67. Current 1 in Problem 3

5. **(*LC*-circuit)** This is an *RLC*-circuit with negligibly small R (analog of an undamped mass–spring system). Find the current when $L = 0.2$ H, $C = 0.05$ F, and $E = \sin t$ V, assuming zero initial current and charge.

6. **(*LC*-circuit)** Find the current when $L = 0.5$ H, $C = 8 \cdot 10^{-4}$ F, $E = t^2$ V and initial current and charge zero.

7–9 *RLC*-CIRCUITS (FIG. 60, P. 92)

7. **(Tuning)** In tuning a stereo system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an *RLC*-circuit so that the amplitude of the steady-state current (5) becomes maximum. For what C will this happen?

8. **(Transient current)** Prove the claim in the text that if $R \neq 0$ (hence $R > 0$), then the transient current approaches I_p as $t \to \infty$.

9. **(Cases of damping)** What are the conditions for an *RLC*-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{mk}$)?

10–12 Find the **steady-state current** in the *RLC*-circuit in Fig. 60 on p. 92 for the given data. (Show the details of your work.)

10. $R = 8\ \Omega$, $L = 0.5$ H, $C = 0.1$ F, $E = 100 \sin 2t$ V

11. $R = 1\ \Omega$, $L = 0.25$ H, $C = 5 \cdot 10^{-5}$ F, $E = 110$ V

12. $R = 2\ \Omega$, $L = 1$ H, $C = 0.05$ F, $E = \frac{157}{9} \sin 3t$ V

13–15 Find the **transient current** (a general solution) in the *RLC*-circuit in Fig. 60 for the given data. (Show the details of your work.)

13. $R = 6\ \Omega$, $L = 0.2$ H, $C = 0.025$ F, $E = 110 \sin 10t$ V

14. $R = 0.2\ \Omega$, $L = 0.1$ H, $C = 2$ F, $E = 754 \sin 0.5t$ V

15. $R = 1/10\ \Omega$, $L = 1/2$ H, $C = 100/13$ F, $E = e^{-4t}(1.932 \cos \frac{1}{2}t + 0.246 \sin \frac{1}{2}t)$ V

16–18 Solve the **initial value problem** for the *RLC*-circuit in Fig. 60 with the given data, assuming zero initial current and charge. Graph or sketch the solution. (Show the details of your work.)

16. $R = 4\ \Omega$, $L = 0.1$ H, $C = 0.025$ F, $E = 10 \sin 10t$ V

17. $R = 6\ \Omega$, $L = 1$ H, $C = 0.04$ F, $E = 600(\cos t + 4 \sin t)$ V

18. $R = 3.6\ \Omega$, $L = 0.2$ H, $C = 0.0625$ F, $E = 164 \cos 10t$ V

19. WRITING PROJECT. Analogy of *RLC*-Circuits and Damped Mass–Spring Systems. **(a)** Write an essay of 2–3 pages based on Table 2.2. Describe the analogy in more detail and indicate its practical significance.

(b) What *RLC*-circuit with $L = 1$ H is the analog of the mass–spring system with mass 5 kg, damping constant 10 kg/sec, spring constant 60 kg/sec^2, and driving force $220 \cos 10t$?

(c) Illustrate the analogy with another example of your own choice.

20. TEAM PROJECT. Complex Method for Particular Solutions. **(a)** Find a particular solution of the **complex ODE**

$$\text{(8)} \qquad L\tilde{I}'' + R\tilde{I}' + \frac{1}{C}\tilde{I} = E_0\omega e^{i\omega t} \qquad (i = \sqrt{-1})$$

by substituting $\tilde{I}_p = Ke^{i\omega t}$ (K unknown) and its derivatives into (8), and then take the real part I_p of $\tilde{I}_p$, showing that I_p agrees with (2), (4). *Hint.* Use the Euler formula $e^{i\omega t} = \cos \omega t + i \sin \omega t$ [(11) in Sec. 2.2 with ωt instead of t]. Note that $E_0\omega \cos \omega t$ in (1) is the real part of $E_0\omega e^{i\omega t}$ in (8). Use $i^2 = -1$.

(b) The **complex impedance** Z is defined by

$$Z = R + iS = R + i\left(\omega L - \frac{1}{\omega C}\right).$$

Show that K obtained in (a) can be written as

$$K = \frac{E_0}{iZ}.$$

Note that the real part of Z is R, the imaginary part is the reactance S, and the absolute value is the impedance $|Z| = \sqrt{R^2 + S^2}$ as defined before. See Fig. 68.

(c) Find the steady-state solution of the ODE $I'' + 2I' + 3I = 20 \cos t$, first by the real method and then by the complex method, and compare. (Show the details of your work.)

(d) Apply the complex method to an *RLC*-circuit of your choice.

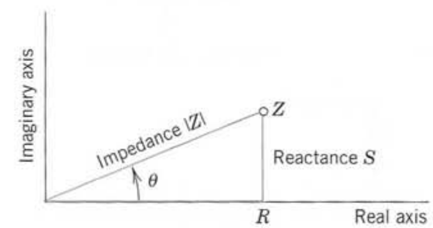


Fig. 68. Complex impedance Z

2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs

$$\text{(1)} \qquad y'' + p(x)y' + q(x)y = r(x).$$

In Sec. 2.6 we have seen that a general solution of (1) is the sum of a general solution y_h of the corresponding homogeneous ODE and any particular solution y_p of (1). To obtain y_p when $r(x)$ is not too complicated, we can often use the *method of undetermined coefficients*, as we have shown in Sec. 2.7 and applied to basic engineering models in Secs. 2.8 and 2.9.

However, since this method is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself (powers, exponential functions, etc.), it is desirable to have a method valid for more general ODEs (1), which we shall now develop. It is called the **method of variation of parameters** and is credited to Lagrange (Sec. 2.1). Here p, q, r in (1) may be variable (given functions of x), but we assume that they are continuous on some open interval I.

Lagrange's method gives a particular solution y_p of (1) on I in the form

$$\text{(2)} \qquad y_p(x) = -y_1\int \frac{y_2 r}{W}\,dx + y_2\int \frac{y_1 r}{W}\,dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

on I, and W is the Wronskian of y_1, y_2,

$$W = y_1 y_2' - y_2 y_1' \qquad \text{(see Sec. 2.6).} \tag{4}$$

CAUTION! The solution formula (2) is obtained under the assumption that the ODE is written in standard form, with y'' as the first term as shown in (1). If it starts with $f(x)y''$, divide first by $f(x)$.

The integration in (2) may often cause difficulties, and so may the determination of y_1, y_2 if (1) has variable coefficients. If you have a choice, use the previous method. It is simpler. Before deriving (2) let us work an example for which you *do need* the new method. (Try otherwise.)

EXAMPLE 1 Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution. A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$\begin{aligned} y_p &= -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx \\ &= \cos x \ln |\cos x| + x \sin x \end{aligned} \qquad \text{(Fig. 69).}$$

Figure 69 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. (Recall from Sec. 2.8 that we have seen $x \sin x$ in connection with resonance, except for notation.) From y_p and the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous ODE we obtain the *answer*

$$y = y_h + y_p = (c_1 + \ln |\cos x|) \cos x + (c_2 + x) \sin x.$$

Had we included integration constants $-c_1$, c_2 in (2), then (2) would have given the additional $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case. ■

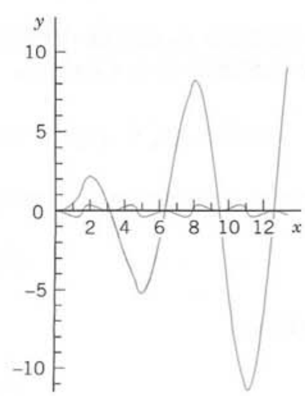


Fig. 69. Particular solution y_p and its first term in Example 1

Idea of the Method. Derivation of (2)

What idea did Lagrange have? What gave the method the name? Where do we use the continuity assumptions?

The idea is to start from a general solution

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

of the homogeneous ODE (3) on an open interval I and to replace the constants ("the parameters") c_1 and c_2 by functions $u(x)$ and $v(x)$; this suggests the name of the method. We shall determine u and v so that the resulting function

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \tag{5}$$

is a particular solution of the nonhomogeneous ODE (1). Note that y_h exists by Theorem 3 in Sec. 2.6 because of the continuity of p and q on I. (The continuity of r will be used later.)

We determine u and v by substituting (5) and its derivatives into (1). Differentiating (5), we obtain

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'.$$

Now y_p must satisfy (1). This is *one* condition for *two* functions u and v. It seems plausible that we may impose a *second* condition. Indeed, our calculation will show that we can determine u and v such that y_p satisfies (1) and u and v satisfy as a second condition the equation

$$u'y_1 + v'y_2 = 0. \tag{6}$$

This reduces the first derivative y_p' to the simpler form

$$y_p' = uy_1' + vy_2'. \tag{7}$$

Differentiating (7), we obtain

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''. \tag{8}$$

We now substitute y_p and its derivatives according to (5), (7), (8) into (1). Collecting terms in u and terms in v, we obtain

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r.$$

Since y_1 and y_2 are solutions of the homogeneous ODE (3), this reduces to

$$u'y_1' + v'y_2' = r. \tag{9a}$$

Equation (6) is

$$u'y_1 + v'y_2 = 0. \tag{9b}$$

This is a linear system of two algebraic equations for the unknown functions u' and v'. We can solve it by elimination as follows (or by Cramer's rule in Sec. 7.6). To eliminate v', we multiply (9a) by $-y_2$ and (9b) by y_2' and add, obtaining

$$u'(y_1y_2' - y_2y_1') = -y_2r, \qquad \text{thus} \qquad u'W = -y_2r.$$

Here, W is the Wronskian (4) of y_1, y_2. To eliminate u' we multiply (9a) by y_1, and (9b) by $-y_1'$ and add, obtaining

$$v'(y_1y_2' - y_2y_1') = y_1r, \qquad \text{thus} \qquad v'W = y_1r.$$

Since y_1, y_2 form a basis, we have $W \neq 0$ (by Theorem 2 in Sec. 2.6) and can divide by W,

$$u' = -\frac{y_2r}{W}, \qquad v' = \frac{y_1r}{W}. \tag{10}$$

By integration,

$$u = -\int \frac{y_2r}{W}\,dx, \qquad v = \int \frac{y_1r}{W}\,dx.$$

These integrals exist because $r(x)$ is continuous. Inserting them into (5) gives (2) and completes the derivation. ■

PROBLEM SET 2.10

1–17 GENERAL SOLUTION

Solve the given nonhomogeneous ODE by variation of parameters or undetermined coefficients. Give a general solution. (Show the details of your work.)

1. $y'' + y = \csc x$
2. $y'' - 4y' + 4y = x^2e^x$
3. $x^2y'' - 2xy' + 2y = x^3 \cos x$
4. $y'' - 2y' + y = e^x \sin x$
5. $y'' + y = \tan x$
6. $x^2y'' - xy' + y = x \ln |x|$
7. $y'' + y = \cos x + \sec x$
8. $y'' - 4y' + 4y = 12e^{2x}/x^4$
9. $(D^2 - 2D + I)y = x^2 + x^{-2}e^x$
10. $(D^2 - I)y = 1/\cosh x$
11. $(D^2 + 4I)y = \cosh 2x$
12. $(x^2D^2 + xD - \frac{1}{4}I)y = 3x^{-1} + 3x$
13. $(x^2D^2 - 2xD + 2I)y = x^3 \sin x$
14. $(x^2D^2 + xD - 4I)y = 1/x^2$
15. $(D^2 + I)y = \sec x - 10 \sin 5x$
16. $(x^2D^2 + xD + (x^2 - \frac{1}{4})I)y = x^{3/2} \cos x$. *Hint.* To find y_1, y_2 set $y = ux^{-1/2}$.
17. $(x^2D^2 + xD + (x^2 - \frac{1}{4})I)y = x^{3/2} \sin x$. *Hint:* As in Prob. 16.

18. **TEAM PROJECT. Comparison of Methods.** The undetermined-coefficient method should be used whenever possible because it is simpler. Compare it with the present method as follows.

    (a) Solve $y'' + 2y' - 15y = 17 \sin 5x$ by both methods, showing all details, and compare.

    (b) Solve $y'' + 9y = r_1 + r_2$, $r_1 = \sec 3x$, $r_2 = \sin 3x$ by applying each method to a suitable function on the right.

    (c) Invent an undetermined-coefficient method for nonhomogeneous Euler–Cauchy equations by experimenting.

CHAPTER 2 REVIEW QUESTIONS AND PROBLEMS

1. What general properties make *linear* ODEs particularly attractive?
2. What is a general solution of a linear ODE? A basis of solutions?
3. How would you obtain a general solution of a nonhomogeneous linear ODE if you knew a general solution of the corresponding homogeneous ODE?
4. What does an initial value problem for a second-order ODE look like?
5. What is a particular solution and why is it more common than a general solution as the answer to practical problems?
6. Why are second-order ODEs more important in modeling than ODEs of higher order?
7. Describe the applications of ODEs in mechanical vibrating systems. What are the electrical analogs of those systems?
8. If a construction, such as a bridge, shows undesirable resonance, what could you do?

9–18 GENERAL SOLUTION

Find a general solution. Indicate the method you are using and show the details of your calculation.

9. $y'' - 2y' - 8y = 52 \cos 6x$
10. $y'' + 6y' + 9y = e^{-3x} - 27x^2$
11. $y'' + 8y' + 25y = 26 \sin 3x$
12. $yy'' = 2y'^2$
13. $(x^2D^2 + 2xD - 12I)y = 1/x^3$
14. $(x^2D^2 + 6xD + 6I)y = x^2$
15. $(D^2 - 2D + I)y = x^{-3}e^x$
16. $(D^2 - 4D + 5I)y = e^{2x} \csc x$
17. $(D^2 - 2D + 2I)y = e^x \csc x$
18. $(4x^2D^2 - 24xD + 49I)y = 36x^5$

19–25 INITIAL VALUE PROBLEMS

Solve the following initial value problems. Sketch or graph the solution. (Show the details of your work.)

19. $y'' + 5y' - 14y = 0, \quad y(0) = 6, \quad y'(0) = -6$
20. $y'' + 6y' + 18y = 0, \quad y(0) = 5, \quad y'(0) = -21$
21. $x^2y'' - xy' - 24y = 0, \quad y(1) = 15, \quad y'(1) = 0$
22. $x^2y'' + 15xy' + 49y = 0, \quad y(1) = 2, \quad y'(1) = -11$
23. $y'' + 5y' + 6y = 108x^2, \quad y(0) = 18, \quad y'(0) = -26$
24. $y'' + y' + 2.5y = 13 \cos x, \; y(0) = 8.0, \; y'(0) = 4.5$
25. $(x^2D^2 + xD - 4I)y = x^3, \; y(1) = -4/5, \; y'(1) = 93/5$

26–34 APPLICATIONS

26. Find the steady-state solution of the system in Fig. 70 when $m = 4$, $c = 4$, $k = 17$ and the driving force is $202 \cos 3t$.
27. Find the motion of the system in Fig. 70 with mass 0.25 kg, no damping, spring constant 1 kg/sec^2, and driving force $15 \cos 0.5t - 7 \sin 1.5t$ nt, assuming zero initial displacement and velocity. For what frequency of the driving force would you get resonance?
28. In Prob. 26 find the solution corresponding to initial displacement 10 and initial velocity 0.
29. Show that the system in Fig. 70 with $m = 4$, $c = 0$, $k = 36$, and driving force $61 \cos 3.1t$ exhibits beats. *Hint:* Choose zero initial conditions.
30. In Fig. 70 let $m = 2$, $c = 6$, $k = 27$, and $r(t) = 10 \cos \omega t$. For what ω will you obtain the steady-state vibration of maximum possible amplitude? Determine this amplitude. Then use this ω and the undetermined-coefficient method to see whether you obtain the same amplitude.
31. Find an electrical analog of the mass–spring system in Fig. 70 with mass 0.5 kg, spring constant 40 kg/sec^2, damping constant 9 kg/sec, and driving force $102 \cos 6t$ nt. Solve the analog, assuming zero initial current and charge.
32. Find the current in the *RLC*-circuit in Fig. 71 when $L = 0.1$ H, $R = 20\ \Omega$, $C = 2 \cdot 10^{-4}$ F, and $E(t) = 110 \sin 415t$ V (66 cycles/sec).
33. Find the current in the *RLC*-circuit when $L = 0.4$ H, $R = 40\ \Omega$, $C = 10^{-4}$ F, and $E(t) = 220 \sin 314t$ V (50 cycles/sec).
34. Find a particular solution in Prob. 33 by the complex method. (See Team Project 20 in Sec. 2.9.)

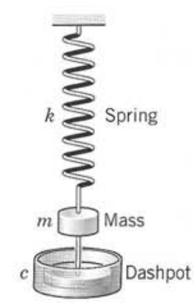


Fig. 70. Mass–spring system

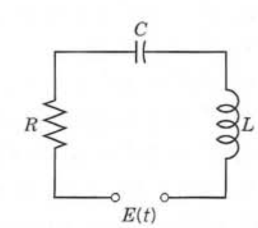


Fig. 71. *RLC*-circuit

SUMMARY OF CHAPTER 2
Second-Order Linear ODEs

Second-order linear ODEs are particularly important in applications, for instance, in mechanics (Secs. 2.4, 2.8) and electrical engineering (Sec. 2.9). A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(Sec. 2.1).} \tag{1}$$

(If the first term is, say, $f(x)y''$, divide by $f(x)$ to get the **"standard form"** (1) with y'' as the first term.) Equation (1) is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval; this is written $r(x) \equiv 0$. Then

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Equation (1) is called **nonhomogeneous** if $r(x) \not\equiv 0$ (meaning $r(x)$ is not zero for some x considered).

For the homogeneous ODE (2) we have the important **superposition principle** (Sec. 2.1) that a linear combination $y = ky_1 + ly_2$ of two solutions y_1, y_2 is again a solution.

Two *linearly independent* solutions y_1, y_2 of (2) on an open interval I form a **basis** (or **fundamental system**) of solutions on I, and $y = c_1y_1 + c_2y_2$ with arbitrary constants c_1, c_2 is a **general solution** of (2) on I. From it we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two **initial conditions**

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \qquad (x_0, K_0, K_1 \text{ given numbers; Sec. 2.1}). \tag{3}$$

(2) and (3) together form an **initial value problem.** Similarly for (1) and (3).

For a nonhomogeneous ODE (1) a **general solution** is of the form

$$y = y_h + y_p \qquad \text{(Sec. 2.7).} \tag{4}$$

Here y_h is a general solution of (2) and y_p is a particular solution of (1). Such a y_p can be determined by a general method (**variation of parameters,** Sec. 2.10) or in many practical cases by the **method of undetermined coefficients.** The latter applies when (1) has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Sec. 2.7). Then we write (1) as

$$y'' + ay' + by = r(x) \qquad \text{(Sec. 2.7).} \tag{5}$$

The corresponding homogeneous ODE $y' + ay' + by = 0$ has solutions $y = e^{\lambda x}$, where λ is a root of

$$\lambda^2 + a\lambda + b = 0. \tag{6}$$

Hence there are three cases (Sec. 2.2):

Case	Type of Roots	General Solution
I	Distinct real λ_1, λ_2	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Double $-\frac{1}{2}a$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex $-\frac{1}{2}a \pm i\omega^*$	$y = e^{-ax/2}(A \cos \omega^* x + B \sin \omega^* x)$

Important applications of (5) in mechanical and electrical engineering in connection with **vibrations** and **resonance** are discussed in Secs. 2.4, 2.7, and 2.8.

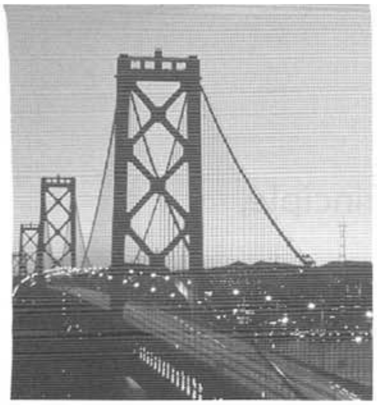
Another large class of ODEs solvable "algebraically" consists of the **Euler–Cauchy equations**

$$x^2 y'' + axy' + by = 0 \qquad (7) \quad \text{(Sec. 2.5).}$$

These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

$$m^2 + (a - 1)m + b = 0. \qquad (8)$$

Existence and uniqueness of solutions of (1) and (2) is discussed in Secs. 2.6 and 2.7, and **reduction of order** in Sec. 2.1.



CHAPTER 3

Higher Order Linear ODEs

In this chapter we extend the concepts and methods of Chap. 2 for linear ODEs from order $n = 2$ to arbitrary order n. This will be straightforward and needs no new ideas. However, the formulas become more involved, the variety of roots of the characteristic equation (in Sec. 3.2) becomes much larger with increasing n, and the Wronskian plays a more prominent role.

Prerequisite: Secs. 2.1, 2.2, 2.6, 2.7, 2.10.
References and Answers to Problems: App. 1 Part A, and App. 2.

3.1 Homogeneous Linear ODEs

Recall from Sec. 1.1 that an ODE is of ***n*th order** if the nth derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Thus the ODE is of the form

$$F(x, y, y', \cdots, y^{(n)}) = 0 \qquad \left(y^{(n)} = \frac{d^n y}{dx^n}\right)$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{1}$$

(For $n = 2$ this is (1) in Sec. 2.1 with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the right are any given functions of x, and y is unknown. $y^{(n)}$ has coefficient 1. This is practical. We call this the **standard form.** (If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.) An nth-order ODE that cannot be written in the form (1) is called **nonlinear.**

If $r(x)$ is identically zero, $r(x) \equiv 0$ (zero for all x considered, usually in some open interval I), then (1) becomes

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{2}$$

and is called **homogeneous.** If $r(x)$ is not identically zero, then the ODE is called **nonhomogeneous.** This is as in Sec. 2.1.

A **solution** of an nth-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that is defined and n times differentiable on I and is such that the ODE becomes an identity if we replace the unknown function y and its derivatives by h and its corresponding derivatives.

Homogeneous Linear ODE: Superposition Principle, General Solution

Sections 3.1–3.2 will be devoted to homogeneous linear ODEs and Sec. 3.3 to nonhomogeneous linear ODEs. The basic **superposition or linearity principle** in Sec. 2.1 extends to nth order homogeneous linear ODEs as follows.

THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *sums and constant multiples of solutions on some open interval I are again solutions on I.* (This does ***not*** hold for a nonhomogeneous or nonlinear ODE!)

The proof is a simple generalization of that in Sec. 2.1 and we leave it to the student.

Our further discussion parallels and extends that for second-order ODEs in Sec. 2.1. So we define next a general solution of (2), which will require an extension of linear independence from 2 to n functions.

DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of (2) on an open interval I is a solution of (2) on I of the form

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \text{ arbitrary}) \tag{3}$$

where $y_1, \cdots, y_n$ is a **basis** (or **fundamental system**) of solutions of (2) on I; that is, these solutions are linearly independent on I, as defined below.

A **particular solution** of (2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in (3).

DEFINITION

Linear Independence and Dependence

n functions $y_1(x), \cdots, y_n(x)$ are called **linearly independent** *on some interval* I where they are defined if the equation

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on } I \tag{4}$$

implies that all $k_1, \cdots, k_n$ are zero. These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

(As in Secs. 1.1 and 2.1, the arbitrary constants $c_1, \cdots, c_n$ must sometimes be restricted to some interval.)

If and only if $y_1, \cdots, y_n$ are linearly dependent on I, we can express (at least) one of these functions on I as a **"linear combination"** of the other $n - 1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not). This motivates the term "linearly dependent." For instance, if (4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Note that when $n = 2$, these concepts reduce to those defined in Sec. 2.1.

EXAMPLE 1 **Linear Dependence**

Show that the functions $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$ are linearly dependent on any interval.

Solution. $y_2 = 0y_1 + 2.5y_3$. This proves linear dependence on any interval. ■

EXAMPLE 2 **Linear Independence**

Show that $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are linearly independent on any interval, for instance, on $-1 \leqq x \leqq 2$.

Solution. Equation (4) is $k_1 x + k_2 x^2 + k_3 x^3 = 0$. Taking (a) $x = -1$, (b) $x = 1$, (c) $x = 2$, we get

$$\text{(a) } -k_1 + k_2 - k_3 = 0, \qquad \text{(b) } k_1 + k_2 + k_3 = 0, \qquad \text{(c) } 2k_1 + 4k_2 + 8k_3 = 0.$$

$k_2 = 0$ from (a) + (b). Then $k_3 = 0$ from (c) -2(b). Then $k_1 = 0$ from (b). This proves linear independence.
A better method for testing linear independence of solutions of ODEs will soon be explained. ■

EXAMPLE 3 **General Solution. Basis**

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad \text{(where } y^{\text{iv}} = d^4y/dx^4\text{)}.$$

Solution. As in Sec. 2.2 we try and substitute $y = e^{\lambda x}$. Omitting the common factor $e^{\lambda x}$, we obtain the characteristic equation

$$\lambda^4 - 5\lambda^2 + 4 = 0.$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0.$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

provided those four solutions are linearly independent. This is true but will be shown later. ■

Initial Value Problem. Existence and Uniqueness

An **initial value problem** for the ODE (2) consists of (2) and n **initial conditions**

$$\textbf{(5)} \qquad y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with given x_0 in the open interval I considered, and given $K_0, \cdots, K_{n-1}$.

In extension of the existence and uniqueness theorem in Sec. 2.6 we now have the following.

THEOREM 2 **Existence and Uniqueness Theorem for Initial Value Problems**

If the coefficients $p_0(x), \cdots, p_{n-1}(x)$ of (2) *are continuous on some open interval I and x_0 is in I, then the initial value problem* (2), (5) *has a unique solution $y(x)$ on I.*

Existence is proved in Ref. [A11] in App. 1. Uniqueness can be proved by a slight generalization of the uniqueness proof at the beginning of App. 4.

EXAMPLE 4 **Initial Value Problem for a Third-Order Euler–Cauchy Equation**

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \qquad y'(1) = 1, \qquad y''(1) = -4.$$

Solution. ***Step 1. General solution.*** As in Sec. 2.5 we try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$, we can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I (see Example 2). [In general one shall need a root-finding method, such as Newton's (Sec. 19.2), also available in a CAS (Computer Algebra System).] Hence a general solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) are not continuous.

Step 2. Particular solution. The derivatives are $y' = c_1 + 2c_2x + 3c_3x^2$ and $y'' = 2c_2 + 6c_3x$. From this and y and the initial conditions we get by setting $x = 1$

$$\begin{aligned} &\text{(a)}\ y(1) = c_1 + c_2 + c_3 = 2 \\ &\text{(b)}\ y'(1) = c_1 + 2c_2 + 3c_3 = 1 \\ &\text{(c)}\ y''(1) = 2c_2 + 6c_3 = -4. \end{aligned}$$

This is solved by Cramer's rule (Sec. 7.6), or by elimination, which is simple, as follows. (b) − (a) gives (d) $c_2 + 2c_3 = -1$. Then (c) − 2(d) gives $c_3 = -1$. Then (c) gives $c_2 = 1$. Finally $c_1 = 2$ from (a).
Answer: $y = 2x + x^2 - x^3$. ■

Linear Independence of Solutions. Wronskian

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. Now Theorem 2 in Sec. 2.6 extends from order $n = 2$ to any n. This extended criterion uses the **Wronskian** W of n solutions $y_1, \cdots, y_n$ defined as the nth order determinant

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}. \tag{6}$$

Note that W depends on x since $y_1, \cdots, y_n$ does. The criterion states that these solutions form a basis if and only if W is not zero; more precisely:

THEOREM 3 **Linear Dependence and Independence of Solutions**

Let the ODE (2) have continuous coefficients $p_0(x), \cdots, p_{n-1}(x)$ on an open interval I. Then n solutions $y_1, \cdots, y_n$ of (2) on I are linearly dependent on I if and only if their Wronskian is zero for some $x = x_0$ in I. Furthermore, if W is zero for $x = x_0$, then W is identically zero on I. Hence if there is an x_1 in I at which W is not zero, then $y_1, \cdots, y_n$ are linearly independent on I, so that they form a basis of solutions of (2) on I.

PROOF **(a)** Let $y_1, \cdots, y_n$ be linearly dependent solutions of (2) on I. Then, by definition, there are constants $k_1, \cdots, k_n$ *not all zero*, such that for all x in I,

$$k_1 y_1 + \cdots + k_n y_n = 0. \tag{7}$$

By $n - 1$ differentiations of (7) we obtain for all x in I

$$\begin{aligned} k_1 y_1' + \cdots + k_n y_n' &= 0 \\ &\vdots \\ k_1 y_1^{(n-1)} + \cdots + k_n y_n^{(n-1)} &= 0. \end{aligned} \tag{8}$$

(7), (8) is a homogeneous linear system of algebraic equations with a nontrivial solution $k_1, \cdots, k_n$. Hence its coefficient determinant must be zero for every x on I, by Cramer's theorem (Sec. 7.7). But that determinant is the Wronskian W, as we see from (6). Hence W is zero for every x on I.

(b) Conversely, if W is zero at an x_0 in I, then the system (7), (8) with $x = x_0$ has a solution $k_1^*, \cdots, k_n^*$, not all zero, by the same theorem. With these constants we define the solution $y^* = k_1^* y_1 + \cdots + k_n^* y_n$ of (2) on I. By (7), (8) this solution satisfies the initial conditions $y^*(x_0) = 0, \cdots, y^{*(n-1)}(x_0) = 0$. But another solution satisfying the same conditions is $y \equiv 0$. Hence $y^* \equiv y$ by Theorem 2, which applies since the coefficients of (2) are continuous. Together, $y^* = k_1^* y_1 + \cdots + k_n^* y_n \equiv 0$ on I. This means linear dependence of $y_1, \cdots, y_n$ on I.

(c) If W is zero at an x_0 in I, we have linear dependence by (b) and then $W \equiv 0$ by (a). Hence if W is not zero at an x_1 in I, the solutions $y_1, \cdots, y_n$ must be linearly independent on I. ■

EXAMPLE 5 **Basis, Wronskian**

We can now prove that in Example 3 we do have a basis. In evaluating W, pull out the exponential functions columnwise. In the result, subtract Column 1 from Columns 2, 3, 4 (without changing Column 1). Then expand by Row 1. In the resulting third-order determinant, subtract Column 1 from Column 2 and expand the result by Row 2:

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ -3 & -3 & 0 \\ 7 & 9 & 16 \end{vmatrix} = 72.$$ ■

A General Solution of (2) Includes All Solutions

Let us first show that general solutions always exist. Indeed, Theorem 3 in Sec. 2.6 extends as follows.

THEOREM 4 **Existence of a General Solution**

If the coefficients $p_0(x), \cdots, p_{n-1}(x)$ of (2) are continuous on some open interval I, then (2) has a general solution on I.

PROOF We choose any fixed x_0 in I. By Theorem 2 the ODE (2) has n solutions $y_1, \cdots, y_n$, where y_j satisfies initial conditions (5) with $K_{j-1} = 1$ and all other K's equal to zero. Their Wronskian at x_0 equals 1. For instance, when $n = 3$, then $y_1(x_0) = 1$, $y_2'(x_0) = 1$, $y_3''(x_0) = 1$, and the other initial values are zero. Thus, as claimed,

$$W(y_1(x_0), y_2(x_0), y_3(x_0)) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & y_3(x_0) \\ y_1'(x_0) & y_2'(x_0) & y_3'(x_0) \\ y_1''(x_0) & y_2''(x_0) & y_3''(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Hence for any n those solutions $y_1, \cdots, y_n$ are linearly independent on I, by Theorem 3. They form a basis on I, and $y = c_1y_1 + \cdots + c_ny_n$ is a general solution of (2) on I. ■

We can now prove the basic property that from a general solution of (2) every solution of (2) can be obtained by choosing suitable values of the arbitrary constants. Hence an nth order ***linear*** ODE has no **singular solutions,** that is, solutions that cannot be obtained from a general solution.

THEOREM 5

General Solution Includes All Solutions

If the ODE (2) *has continuous coefficients $p_0(x), \cdots, p_{n-1}(x)$ on some open interval I, then every solution $y = Y(x)$ of* (2) *on I is of the form*

$$Y(x) = C_1y_1(x) + \cdots + C_ny_n(x) \tag{9}$$

where $y_1, \cdots, y_n$ is a basis of solutions of (2) *on I and $C_1, \cdots, C_n$ are suitable constants.*

PROOF Let Y be a given solution and $y = c_1y_1 + \cdots + c_ny_n$ a general solution of (2) on I. We choose any fixed x_0 in I and show that we can find constants $c_1, \cdots, c_n$ for which y and its first $n - 1$ derivatives agree with Y and its corresponding derivatives at x_0. That is, we should have at $x = x_0$

$$\begin{aligned} c_1y_1 + \cdots + \quad c_ny_n \quad &= Y \\ c_1y_1' + \cdots + \quad c_ny_n' \quad &= Y' \\ &\vdots \\ c_1y_1^{(n-1)} + \cdots + c_ny_n^{(n-1)} &= Y^{(n-1)}. \end{aligned} \tag{10}$$

But this is a linear system of equations in the unknowns $c_1, \cdots, c_n$. Its coefficient determinant is the Wronskian W of $y_1, \cdots, y_n$ at x_0. Since $y_1, \cdots, y_n$ form a basis, they are linearly independent, so that W is not zero by Theorem 3. Hence (10) has a unique solution $c_1 = C_1, \cdots, c_n = C_n$ (by Cramer's theorem in Sec. 7.7). With these values we obtain the particular solution

$$y^*(x) = C_1y_1(x) + \cdots + C_ny_n(x)$$

on I. Equation (10) shows that y^* and its first $n - 1$ derivatives agree at x_0 with Y and its corresponding derivatives. That is, y^* and Y satisfy at x_0 the same initial conditions.

The uniqueness theorem (Theorem 2) now implies that $y^* \equiv Y$ on I. This proves the theorem. ■

This completes our theory of the homogeneous linear ODE (2). Note that for $n = 2$ it is identical with that in Sec. 2.6. This had to be expected.

PROBLEM SET 3.1

1–5 TYPICAL EXAMPLES OF BASES

To get a feel for higher order ODEs, show that the given functions are solutions and form a basis on any interval. Use Wronskians. (In Prob. 2, $x > 0$.)

1. $1, x, x^2, x^3, \quad y^{\text{iv}} = 0$
2. $1, x^2, x^4, \quad x^2y''' - 3xy'' + 3y' = 0$
3. $e^x, xe^x, x^2e^x, \quad y''' - 3y'' + 3y' - y = 0$
4. $e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x,$ $y^{\text{iv}} - 6y'' + 25y = 0.$
5. $1, x, \cos 3x, \sin 3x, \quad y^{\text{iv}} + 9y'' = 0$
6. **TEAM PROJECT. General Properties of Solutions of Linear ODEs.** These properties are important in obtaining new solutions from given ones. Therefore extend Team Project 34 in Sec. 2.2 to nth-order ODEs. Explore statements on sums and multiples of solutions of (1) and (2) systematically and with proofs. Recognize clearly that no new ideas are needed in this extension from $n = 2$ to general n.

7–19 LINEAR INDEPENDENCE AND DEPENDENCE

Are the given functions linearly independent or dependent on the positive x-axis? (Give a reason.)

7. $1, e^x, e^{-x}$
8. $x + 1, x + 2, x$
9. $\ln x, \ln x^2, (\ln x)^2$
10. $e^x, e^{-x}, \sinh 2x$
11. $x^2, x|x|, x$
12. $x, 1/x, 0$
13. $\sin 2x, \sin x, \cos x$
14. $\cos^2 x, \sin^2 x, \cos 2x$
15. $\tan x, \cot x, 1$
16. $(x - 1)^2, (x + 1)^2, x$
17. $\sin x, \sin \frac{1}{2}x$
18. $\cosh x, \sinh x, \cosh^2 x$
19. $\cos^2 x, \sin^2 x, 2\pi$
20. **TEAM PROJECT. Linear Independence and Dependence.** **(a)** Investigate the given question about a set S of functions on an interval I. Give an example. Prove your answer.

    (1) If S contains the zero function, can S be linearly independent?

    (2) If S is linearly independent on a subinterval J of I, is it linearly independent on I?

    (3) If S is linearly dependent on a subinterval J of I, is it linearly dependent on I?

    (4) If S is linearly independent on I, is it linearly independent on a subinterval J?

    (5) If S is linearly dependent on I, is it linearly independent on a subinterval J?

    (6) If S is linearly dependent on I, and if T contains S, is T linearly dependent on I?

    (b) In what cases can you use the Wronskian for testing linear independence? By what other means can you perform such a test?

3.2 Homogeneous Linear ODEs with Constant Coefficients

In this section we consider nth-order homogeneous linear ODEs with constant coefficients, which we write in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \tag{1}$$

where $y^{(n)} = d^ny/dx^n$, etc. We shall see that this extends the case $n = 2$ discussed in Sec. 2.2. Substituting $y = e^{\lambda x}$ (as in Sec. 2.2), we obtain the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 = 0 \tag{2}$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We shall discuss all of them and illustrate them with typical examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \quad \cdots, \quad y_n = e^{\lambda_n x} \tag{3}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{4}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

EXAMPLE 1 **Distinct Real Roots**

Solve the ODE $y''' - 2y'' - y' + 2y = 0$.

Solution. The characteristic equation is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$. It has the roots $-1, 1, 2$; if you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation (explain!). The corresponding general solution (4) is $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$. ■

Linear Independence of (3). Students familiar with nth-order determinants may verify that by pulling out all exponential functions from the columns and denoting their product by E, thus $E = \exp[(\lambda_1 + \cdots + \lambda_n)x]$, the Wronskian of the solutions in (3) becomes

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \lambda_1{}^2 e^{\lambda_1 x} & \lambda_2{}^2 e^{\lambda_2 x} & \cdots & \lambda_n{}^2 e^{\lambda_n x} \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= E \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1{}^2 & \lambda_2{}^2 & \cdots & \lambda_n{}^2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}. \tag{5}$$

The exponential function E is never zero. Hence $W = 0$ if and only if the determinant on the right is zero. This is a so-called **Vandermonde** or **Cauchy determinant**[1]. It can be shown that it equals

[1]ALEXANDRE THÉOPHILE VANDERMONDE (1735–1796), French mathematician, who worked on solution of equations by determinants. For CAUCHY see footnote 4, in Sec. 2.5.

$$(-1)^{n(n-1)/2}V \tag{6}$$

where V is the product of all factors $\lambda_j - \lambda_k$ with $j < k\ (\leqq n)$; for instance, when $n = 3$ we get $-V = -(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. This shows that the Wronskian is not zero if and only if all the n roots of (2) are different and thus gives the following.

THEOREM 1

Basis

Solutions $y_1 = e^{\lambda_1 x}, \cdots, y_n = e^{\lambda_n x}$ of (1) (with any real or complex λ_j's) form a basis of solutions of (1) on any open interval if and only if all n roots of (2) are different.

Actually, Theorem 1 is an important special case of our more general result obtained from (5) and (6):

THEOREM 2

Linear Independence

Any number of solutions of (1) of the form $e^{\lambda x}$ are linearly independent on an open interval I if and only if the corresponding λ are all different.

Simple Complex Roots

If complex roots occur, they must occur in conjugate pairs since the coefficients of (1) are real. Thus, if $\lambda = \gamma + i\omega$ is a simple root of (2), so is the conjugate $\bar{\lambda} = \gamma - i\omega$, and two corresponding linearly independent solutions are (as in Sec. 2.2, except for notation)

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

EXAMPLE 2 **Simple Complex Roots. Initial Value Problem**

Solve the initial value problem

$$y''' - y'' + 100y' - 100y = 0, \qquad y(0) = 4, \qquad y'(0) = 11, \qquad y''(0) = -299.$$

Solution. The characteristic equation is $\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$. It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10i$. Hence a general solution and its derivatives (obtained by differentiation) are

$$\begin{aligned} y &= c_1 e^x + A \cos 10x + B \sin 10x, \\ y' &= c_1 e^x - 10A \sin 10x + 10B \cos 10x, \\ y'' &= c_1 e^x - 100A \cos 10x - 100B \sin 10x. \end{aligned}$$

From this and the initial conditions we obtain by setting $x = 0$

$$\text{(a)}\quad c_1 + A = 4, \qquad \text{(b)}\quad c_1 + 10B = 11, \qquad \text{(c)}\quad c_1 - 100A = -299.$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is (Fig. 72)

$$y = e^x + 3 \cos 10x + \sin 10x.$$

This gives the solution curve, which oscillates about e^x (dashed in Fig. 72 on p. 114). ■

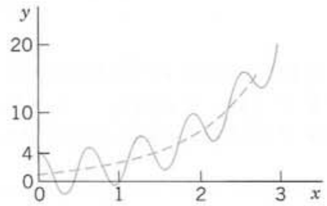


Fig. 72. Solution in Example 2

Multiple Real Roots

If a real double root occurs, say, $\lambda_1 = \lambda_2$, then $y_1 = y_2$ in (3), and we take y_1 and xy_1 as corresponding linearly independent solutions. This is as in Sec. 2.2.

More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2e^{\lambda x}, \quad \cdots, \quad x^{m-1}e^{\lambda x}. \tag{7}$$

We derive these solutions after the next example and indicate how to prove their linear independence.

EXAMPLE 3 Real Double and Triple Roots

Solve the ODE $y^v - 3y^{iv} + 3y''' - y'' = 0$.

Solution. The characteristic equation $\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$ has the roots $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 1$, and the answer is

$$y = c_1 + c_2x + (c_3 + c_4x + c_5x^2)e^x. \tag{8}$$ ■

***Derivation of* (7).** We write the left side of (1) as

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y.$$

Let $y = e^{\lambda x}$. Then by performing the differentiations we have

$$L[e^{\lambda x}] = (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)e^{\lambda x}.$$

Now let λ_1 be a root of mth order of the polynomial on the right, where $m \leqq n$. For $m < n$ let $\lambda_{m+1}, \cdots, \lambda_n$ be the other roots, all different from λ_1. Writing the polynomial in product form, we then have

$$L[e^{\lambda x}] = (\lambda - \lambda_1)^m h(\lambda)e^{\lambda x}$$

with $h(\lambda) = 1$ if $m = n$, and $h(\lambda) = (\lambda - \lambda_{m+1}) \cdots (\lambda - \lambda_n)$ if $m < n$. Now comes the key idea: We differentiate on both sides with respect to λ,

$$\frac{\partial}{\partial\lambda} L[e^{\lambda x}] = m(\lambda - \lambda_1)^{m-1}h(\lambda)e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial\lambda} [h(\lambda)e^{\lambda x}]. \tag{9}$$

The differentiations with respect to x and λ are independent and the occurring derivatives are continuous, so that we can interchange their order on the left:

$$\frac{\partial}{\partial \lambda} L[e^{\lambda x}] = L\left[\frac{\partial}{\partial \lambda} e^{\lambda x}\right] = L[xe^{\lambda x}]. \tag{10}$$

The right side of (9) is zero for $\lambda = \lambda_1$ because of the factors $\lambda - \lambda_1$ (and $m \geqq 2$ since we have a multiple root!). Hence $L[xe^{\lambda_1 x}] = 0$ by (9) and (10). This proves that $xe^{\lambda_1 x}$ is a solution of (1).

We can repeat this step and produce $x^2 e^{\lambda_1 x}, \cdots, x^{m-1}e^{\lambda_1 x}$ by another $m - 2$ such differentiations with respect to λ. Going one step further would no longer give zero on the right because the lowest power of $\lambda - \lambda_1$ would then be $(\lambda - \lambda_1)^0$, multiplied by $m!h(\lambda)$ and $h(\lambda_1) \neq 0$ because $h(\lambda)$ has no factors $\lambda - \lambda_1$; so we get *precisely* the solutions in (7).

We finally show that the solutions (7) are linearly independent. For a specific n this can be seen by calculating their Wronskian, which turns out to be nonzero. For arbitrary m we can pull out the exponential functions from the Wronskian. This gives $(e^{\lambda x})^m = e^{\lambda m x}$ times a determinant which by "row operations" can be reduced to the Wronskian of $1, x, \cdots, x^{m-1}$. The latter is constant and different from zero (equal to $1!2! \cdots (m-1)!$). These functions are solutions of the ODE $y^{(m)} = 0$, so that linear independence follows from Theroem 3 in Sec. 3.1. ■

Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots above. Consequently, if $\lambda = \gamma + i\omega$ is a **complex double root,** so is the conjugate $\bar{\lambda} = \gamma - i\omega$. Corresponding linearly independent solutions are

$$e^{\gamma x}\cos \omega x, \qquad e^{\gamma x}\sin \omega x, \qquad xe^{\gamma x}\cos \omega x, \qquad xe^{\gamma x}\sin \omega x. \tag{11}$$

The first two of these result from $e^{\lambda x}$ and $e^{\bar{\lambda} x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\bar{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

$$y = e^{\gamma x}[(A_1 + A_2 x)\cos \omega x + (B_1 + B_2 x)\sin \omega x]. \tag{12}$$

For *complex triple roots* (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\lambda x}\cos \omega x$, $x^2 e^{\gamma x}\sin \omega x$, and so on.

PROBLEM SET 3.2

1–6 ODE FOR GIVEN BASIS

Find an ODE (1) for which the given functions form a basis of solutions.

1. e^x, e^{2x}, e^{3x}

2. $e^{-2x}, xe^{-2x}, x^2e^{-2x}$

3. $e^x, e^{-x}, \cos x, \sin x$

4. $\cos x, \sin x, x\cos x, x\sin x$

5. $1, x, \cos 2x, \sin 2x$

6. $e^{-2x}, e^{-x}, e^x, e^{2x}, 1$

7–12 GENERAL SOLUTION

Solve the given ODE. (Show the details of your work.)

7. $y''' + y' = 0$

8. $y^{\text{iv}} - 29y'' + 100y = 0$

9. $y''' + y'' - y' - y = 0$

10. $16y^{\text{iv}} - 8y'' + y = 0$

11. $y''' - 3y'' - 4y' + 6y = 0$

12. $y^{\text{iv}} + 3y'' - 4y = 0$

13–18 INITIAL VALUE PROBLEMS

Solve by a CAS, giving a general solution and the particular solution and its graph.

13. $y^{\text{iv}} + 0.45y''' - 0.165y'' + 0.0045y' - 0.00175y = 0$, $y(0) = 17.4$, $y'(0) = -2.82$, $y''(0) = 2.0485$, $y'''(0) = -1.458675$

14. $4y''' + 8y'' + 41y' + 37y = 0$, $y(0) = 9$, $y'(0) = -6.5$, $y''(0) = -39.75$

15. $y''' + 3.2y'' + 4.81y' = 0$, $y(0) = 3.4$, $y'(0) = -4.6$, $y''(0) = 9.91$

16. $y^{\text{iv}} + 4y = 0$, $y(0) = \frac{1}{2}$, $y'(0) = -\frac{3}{2}$, $y''(0) = \frac{5}{2}$, $y'''(0) = -\frac{7}{2}$

17. $y^{\text{iv}} - 9y'' - 400y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 41$, $y'''(0) = 0$

18. $y''' + 7.5y'' + 14.25y' - 9.125y = 0$, $y(0) = 10.05$, $y'(0) = -54.975$, $y''(0) = 257.5125$

19. CAS PROJECT. Wronskians. Euler–Cauchy Equations of Higher Order. Although Euler–Cauchy equations have *variable* coefficients (powers of x), we include them here because they fit quite well into the present methods.

(a) Write a program for calculating Wronskians.

(b) Apply the program to some bases of third-order and fourth-order constant-coefficient ODEs. Compare the results with those obtained by the program most likely available for Wronskians in your CAS.

(c) Extend the solution method in Sec. 2.5 to any order n. Solve $x^3y''' + 2x^2y'' - 4xy' + 4y = 0$ and another ODE of your choice. In each case calculate the Wronskian.

20. PROJECT. Reduction of Order. This is of practical interest since a single solution of an ODE can often be guessed. For second order, see Example 7 in Sec. 2.1.

(a) How could you reduce the order of a linear constant-coefficient ODE if a solution is known?

(b) Extend the method to a variable-coefficient ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

Assuming a solution y_1 to be known, show that another solution is $y_2(x) = u(x)y_1(x)$ with $u(x) = \int z(x)\,dx$ and z obtained by solving

$$y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0.$$

(c) Reduce

$$x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0,$$

using $y_1 = x$ (perhaps obtainable by inspection).

21. CAS EXPERIMENT. Reduction of Order. Starting with a basis, find third-order ODEs with variable coefficients for which the reduction to second order turns out to be relatively simple.

3.3 Nonhomogeneous Linear ODEs

We now turn from homogeneous to nonhomogeneous linear ODEs of nth order. We write them in standard form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{1}$$

with $y^{(n)} = d^ny/dx^n$ as the first term, which is practical, and $r(x) \not\equiv 0$. As for second-order ODEs, a general solution of (1) on an open interval I of the x-axis is of the form

$$y(x) = y_h(x) + y_p(x). \tag{2}$$

Here $y_h(x) = c_1y_1(x) + \cdots + c_ny_n(x)$ is a general solution of the corresponding homogeneous ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3}$$

on I. Also, y_p is any solution of (1) on I containing no arbitrary constants. If (1) has continuous coefficients and a continuous $r(x)$ on I, then a general solution of (1) exists and includes all solutions. Thus (1) has no singular solutions.

An **initial value problem** for (1) consists of (1) and n **initial conditions**

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \cdots, \quad y^{(n-1)}(x_0) = K_{n-1} \tag{4}$$

with x_0 in I. Under those continuity assumptions it has a unique solution. The ideas of proof are the same as those for $n = 2$ in Sec. 2.7.

Method of Undetermined Coefficients

Equation (2) shows that for solving (1) we have to determine a particular solution of (1). For a constant-coefficient equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x) \tag{5}$$

($a_0, \cdots, a_{n-1}$ constant) and special $r(x)$ as in Sec. 2.7, such a $y_p(x)$ can be determined by the **method of undetermined coefficients,** as in Sec. 2.7, using the following rules.

(A) Basic Rule as in Sec. 2.7.

(B) Modification Rule. *If a term in your choice for $y_p(x)$ is a solution of the homogeneous equation* (3), *then multiply $y_p(x)$ by x^k, where k is the smallest positive integer such that no term of $x^k y_p(x)$ is a solution of* (3).

(C) Sum Rule as in Sec. 2.7.

The practical application of the method is the same as that in Sec. 2.7. It suffices to illustrate the typical steps of solving an initial value problem and, in particular, the new Modification Rule, which includes the old Modification Rule as a particular case (with $k = 1$ or 2). We shall see that the technicalities are the same as for $n = 2$, perhaps except for the more involved determination of the constants.

EXAMPLE 1 **Initial Value Problem. Modification Rule**

Solve the initial value problem

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47. \tag{6}$$

Solution. *Step 1.* The characteristic equation is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$. It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is

$$\begin{aligned} y_h &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \\ &= (c_1 + c_2 x + c_3 x^2)e^{-x}. \end{aligned}$$

Step 2. If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has no solution. Try Cxe^{-x} and Cx^2e^{-x}. The Modification Rule calls for

$$y_p = Cx^3 e^{-x}.$$

Then

$$\begin{aligned} y_p' &= C(3x^2 - x^3)e^{-x}, \\ y_p'' &= C(6x - 6x^2 + x^3)e^{-x}, \\ y_p''' &= C(6 - 18x + 9x^2 - x^3)e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$. This gives $y_p = 5x^3e^{-x}$.

Step 3. We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y''(0)$ and the third initial condition:

$$
\begin{aligned}
y &= y_h + y_p = (c_1 + c_2x + c_3x^2)e^{-x} + 5x^3e^{-x}, && y(0) = c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, && y'(0) = -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, && y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}
$$

Hence the *answer* to our problem is (Fig. 73)

$$y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}.$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$. The dashed curve in Fig. 73 is y_p. ■

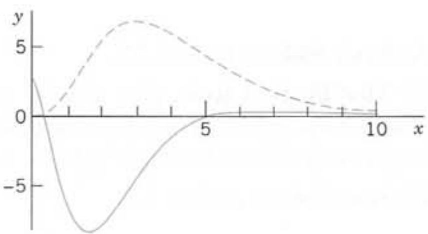


Fig. 73. y and y_p (dashed) in Example 1

Method of Variation of Parameters

The method of variation of parameters (see Sec. 2.10) also extends to arbitrary order n. It gives a particular solution y_p for the nonhomogeneous equation (1) (in standard form with $y^{(n)}$ as the first term!) by the formula

$$
\begin{aligned}
y_p(x) &= \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)}\, r(x)\, dx \\
&= y_1(x) \int \frac{W_1(x)}{W(x)}\, r(x)\, dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)}\, r(x)\, dx
\end{aligned}
\tag{7}
$$

on an open interval I on which the coefficients of (1) and $r(x)$ are continuous. In (7) the functions $y_1, \cdots, y_n$ form a basis of the homogeneous ODE (3), with Wronskian W, and W_j $(j = 1, \cdots, n)$ is obtained from W by replacing the jth column of W by the column $[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]^{\mathsf{T}}$. Thus, when $n = 2$, this becomes identical with (2) in Sec. 2.10,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1.$$

The proof of (7) uses an extension of the idea of the proof of (2) in Sec. 2.10 and can be found in Ref [A11] listed in App. 1.

EXAMPLE 2 Variation of Parameters. Nonhomogeneous Euler–Cauchy Equation

Solve the nonhomogeneous Euler–Cauchy equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x \qquad (x > 0).$$

Solution. *Step 1. General solution of the homogeneous ODE.* Substitution of $y = x^m$ and the derivatives into the homogeneous ODE and deletion of the factor x^m gives

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0.$$

The roots are 1, 2, 3 and give as a basis

$$y_1 = x, \qquad y_2 = x^2, \qquad y_3 = x^3.$$

Hence the corresponding general solution of the homogeneous ODE is

$$y_h = c_1x + c_2x^2 + c_3x^3.$$

Step 2. Determinants needed in (7). These are

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Step 3. Integration. In (7) we also need the right side $r(x)$ of our ODE in standard form, obtained by division of the given equation by the coefficient x^3 of y'''; thus, $r(x) = (x^4 \ln x)/x^3 = x \ln x$. In (7) we have the simple quotients $W_1/W = x/2$, $W_2/W = -1$, $W_3/W = 1/(2x)$. Hence (7) becomes

$$\begin{aligned} y_p &= x \int \frac{x}{2}\, x \ln x \, dx - x^2 \int x \ln x \, dx + x^3 \int \frac{1}{2x}\, x \ln x \, dx \\ &= \frac{x}{2}\left(\frac{x^3}{3} \ln x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x \ln x - x). \end{aligned}$$

Simplification gives $y_p = \frac{1}{6}x^4 (\ln x - \frac{11}{6})$. Hence the answer is

$$y = y_h + y_p = c_1x + c_2x^2 + c_3x^3 + \tfrac{1}{6}x^4 (\ln x - \tfrac{11}{6}).$$

Figure 74 shows y_p. Can you explain the shape of this curve? Its behavior near $x = 0$? The occurrence of a minimum? Its rapid increase? Why would the method of undetermined coefficients not have given the solution? ■

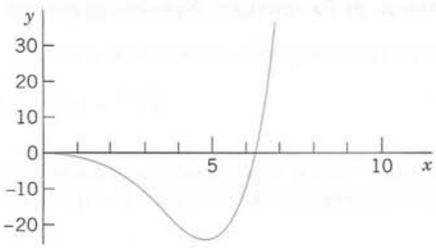


Fig. 74. Particular solution y_p of the nonhomogeneous Euler–Cauchy equation in Example 2

Application: Elastic Beams

Whereas second-order ODEs have various applications, some of the more important ones we have seen, higher order ODEs occur much more rarely in engineering work. An important fourth-order ODE governs the bending of elastic beams, such as wooden or iron girders in a building or a bridge.

Vibrations of beams will be considered in Sec. 12.3.

EXAMPLE 3 Bending of an Elastic Beam under a Load

We consider a beam B of length L and constant (e.g., **rectangular**) cross section and homogeneous elastic material (e.g., steel); see Fig. 75. We assume that under its own weight the beam is bent so little that it is practically straight. If we apply a load to B in a vertical plane through the axis of symmetry (the x-axis in Fig. 75), B is bent. Its axis is curved into the so-called **elastic curve** C (or **deflection curve**). It is shown in elasticity theory that the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. We assume the bending to be small, so that the deflection $y(x)$ and its derivative $y'(x)$ (determining the tangent direction of C) are small. Then, by calculus, $k = y''/(1 + y'^2)^{3/2} \approx y''$. Hence

$$M(x) = EIy''(x).$$

EI is the constant of proportionality. E is *Young's modulus of elasticity* of the material of the beam. I is the moment of inertia of the cross section about the (horizontal) z-axis in Fig. 75.

Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$EIy^{\text{iv}} = f(x). \tag{8}$$

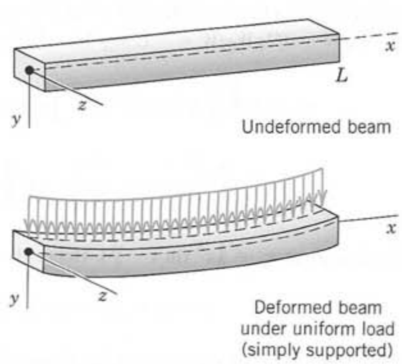


Fig. 75. Elastic Beam

The practically most important supports and corresponding boundary conditions are as follows (see Fig. 76).

(A) Simply supported $y = y'' = 0$ at $x = 0$ and L

(B) Clamped at both ends $y = y' = 0$ at $x = 0$ and L

(C) Clamped at $x = 0$, free at $x = L$ $y(0) = y'(0) = 0,\ y''(L) = y'''(L) = 0.$

The boundary condition $y = 0$ means no displacement at that point, $y' = 0$ means a horizontal tangent, $y'' = 0$ means no bending moment, and $y''' = 0$ means no shear force.

Let us apply this to the uniformly loaded simply supported beam in Fig. 75. The load is $f(x) \equiv f_0 = const.$ Then (8) is

$$y^{\text{iv}} = k, \qquad k = \frac{f_0}{EI}. \tag{9}$$

This can be solved simply by calculus. Two integrations give

$$y'' = \frac{k}{2} x^2 + c_1 x + c_2.$$

$y''(0) = 0$ gives $c_2 = 0$. Then $y''(L) = L(\frac{1}{2}kL + c_1) = 0$, $c_1 = -kL/2$ (since $L \neq 0$). Hence

$$y'' = \frac{k}{2}(x^2 - Lx).$$

Integrating this twice, we obtain

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

with $c_4 = 0$ from $y(0) = 0$. Then

$$y(L) = \frac{kL}{2}\left(\frac{L^3}{12} - \frac{L^3}{6} + c_3\right) = 0, \qquad c_3 = \frac{L^3}{12}.$$

Inserting the expression for k, we obtain as our solution

$$y = \frac{f_0}{24EI}(x^4 - 2Lx^3 + L^3 x).$$

Since the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be "symmetric" with respect to $L/2$, that is, $y(x) = y(L - x)$. Verify this directly or set $x = u + L/2$ and show that y becomes an even function of u,

$$y = \frac{f_0}{24EI}\left(u^2 - \frac{1}{4}L^2\right)\left(u^2 - \frac{5}{4}L^2\right).$$

From this we can see that the maximum deflection in the middle at $u = 0$ $(x = L/2)$ is $5f_0L^4/(16 \cdot 24EI)$. Recall that the positive direction points downward. ■

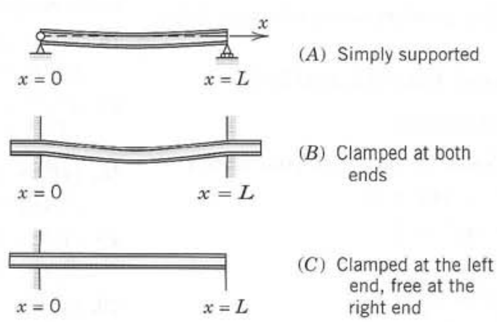


Fig. 76. Supports of a Beam

PROBLEM SET 3.3

1–8 GENERAL SOLUTION

Solve the following ODEs. (Show the details of your work.)

1. $y''' - 2y'' - 4y' + 8y = e^{-3x} + 8x^2$
2. $y''' + 3y'' - 5y' - 39y = 30 \cos x$
3. $y^{\text{iv}} + 0.5y'' + 0.0625y = e^{-x} \cos 0.5x$
4. $y''' + 2y'' - 5y' - 6y = 100e^{-3x} + 18e^{-x}$
5. $x^3y''' + 0.75xy' - 0.75y = 9x^{5.5}$
6. $(xD^3 + 4D^2)y = 8e^x$
7. $(D^4 + 10D^2 + 9I)y = 13 \cosh 2x$
8. $(D^3 - 2D^2 - 9D + 18I)y = e^{2x}$

9–14 INITIAL VALUE PROBLEMS

Solve the following initial value problems. (Show the details.)

9. $y''' - 9y'' + 27y' - 27y = 54 \sin 3x$, $y(0) = 3.5$, $y'(0) = 13.5$, $y''(0) = 38.5$
10. $y^{\text{iv}} - 16y = 128 \cosh 2x$, $y(0) = 1$, $y'(0) = 24$, $y''(0) = 20$, $y'''(0) = -160$
11. $(x^3D^3 - x^2D^2 - 7xD + 16I)y = 9x \ln x$, $y(1) = 6$, $Dy(1) = 18$, $D^2y(1) = 65$
12. $(D^4 - 26D^2 + 25I)y = 50(x + 1)^2$, $y(0) = 12.16$, $Dy(0) = -6$, $D^2y(0) = 34$, $D^3y(0) = -130$
13. $(D^3 + 4D^2 + 85D)y = 135xe^x$, $y(0) = 10.4$, $Dy(0) = -18.1$, $D^2y(0) = -691.6$
14. $(2D^3 - D^2 - 8D + 4I)y = \sin x$, $y(0) = 1$, $Dy(0) = 0$, $D^2y(0) = 0$

15. **WRITING PROJECT. Comparison of Methods.** Write a report on the method of undetermined coefficients and the method of variation of parameters, discussing and comparing the advantages and disadvantages of each method. Illustrate your findings with typical examples. Try to show that the method of undetermined coefficients, say, for a third-order ODE with constant coefficients and an exponential function on the right, can be derived from the method of variation of parameters.

16. **CAS EXPERIMENT. Undetermined Coefficients.** Since variation of parameters is generally complicated, it seems worthwhile to try to extend the other method. Find out experimentally for what ODEs this is possible and for what not. *Hint:* Work backward, solving ODEs with a CAS and then looking whether the solution could be obtained by undetermined coefficients. For example, consider

$$y''' - 12y'' + 48y' - 64y = x^{1/2}e^{4x} \quad \text{and}$$
$$x^3y''' + x^2y'' - 6xy' + 6y = x \ln x.$$

CHAPTER 3 REVIEW QUESTIONS AND PROBLEMS

1. What is the superposition or linearity principle? For what nth-order ODEs does it hold?
2. List some other basic theorems that extend from second-order to nth-order ODEs.
3. If you know a general solution of a homogeneous linear ODE, what do you need to obtain from it a general solution of a corresponding nonhomogeneous linear ODE?
4. What is an initial value problem for an nth-order linear ODE?
5. What is the Wronskian? What is it used for?

6–15 GENERAL SOLUTION

Solve the given ODE. (Show the details of your work.)

6. $y''' + 6y'' + 18y' + 40y = 0$
7. $4x^2y''' + 12xy'' + 3y' = 0$
8. $y^{\text{iv}} + 10y'' + 9y = 0$
9. $8y''' + 12y'' - 2y' - 3y = 0$
10. $(D^3 + 3D^2 + 3D + I)y = x^2$
11. $(xD^4 + D^3)y = 150x^4$
12. $(D^4 - 2D^3 - 8D^2)y = 16 \cos 2x$
13. $(D^3 + I)y = 9e^{x/2}$
14. $(x^3D^3 - 3x^2D^2 + 6xD - 6I)y = 30x^{-2}$
15. $(D^3 - D^2 - D + I)y = e^x$

16–20 INITIAL VALUE PROBLEMS

Solve the given problem. (Show the details.)

16. $y''' - 2y'' + 4y' - 8y = 0$, $y(0) = -1$, $y'(0) = 30$, $y''(0) = 28$
17. $x^3y''' + 7x^2y'' - 2xy' - 10y = 0$, $y(1) = 1$, $y'(1) = -7$, $y''(1) = 44$
18. $(D^3 + 25D)y = 32 \cos^2 4x$, $y(0) = 0$, $Dy(0) = 0$, $D^2y(0) = 0$
19. $(D^4 + 40D^2 - 441I)y = 8 \cosh x$, $y(0) = 1.98$, $Dy(0) = 3$, $D^2y(0) = -40.02$, $D^3y(0) = 27$
20. $(x^3D^3 + 5x^2D^2 + 2xD - 2I)y = 7x^{3/2}$, $y(1) = 10.6$, $Dy(1) = -3.6$, $D^2y(1) = 31.2$

SUMMARY OF CHAPTER 3
Higher Order Linear ODEs

Compare with the similar Summary of Chap. 2 (the case n = 2).
Chapter 3 extends Chap. 2 from order $n = 2$ to arbitrary order n. An **nth-order linear ODE** is an ODE that can be written

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{1}$$

with $y^{(n)} = d^n y/dx^n$ as the first term; we again call this the **standard form.** Equation (1) is called **homogeneous** if $r(x) \equiv 0$ on a given open interval I considered, **nonhomogeneous** if $r(x) \not\equiv 0$ on I. For the homogeneous ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{2}$$

the **superposition principle** (Sec. 3.1) holds, just as in the case $n = 2$. A **basis** or **fundamental system** of solutions of (2) on I consists of n linearly independent solutions $y_1, \cdots, y_n$ of (2) on I. A **general solution** of (2) on I is a linear combination of these,

$$y = c_1 y_1 + \cdots + c_n y_n \qquad (c_1, \cdots, c_n \text{ arbitrary constants}). \tag{3}$$

A **general solution** of the nonhomogeneous ODE (1) on I is of the form

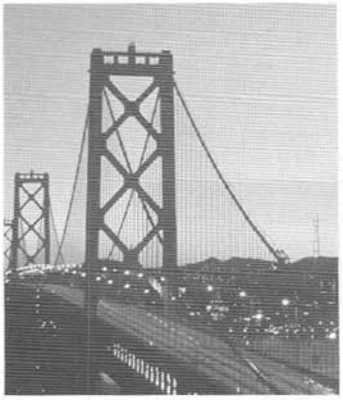
$$y = y_h + y_p \qquad \text{(Sec. 3.3)}. \tag{4}$$

Here, y_p is a particular solution of (1) and is obtained by two methods (**undetermined coefficients** or **variation of parameters**) explained in Sec. 3.3.

An **initial value problem** for (1) or (2) consists of one of these ODEs and n initial conditions (Secs. 3.1, 3.3)

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \cdots, \quad y^{(n-1)}(x_0) = K_{n-1} \tag{5}$$

with given x_0 in I and given $K_0, \cdots, K_{n-1}$. If $p_0, \cdots, p_{n-1}, r$ are continuous on I, then general solutions of (1) and (2) on I exist, and initial value problems (1), (5) or (2), (5) have a unique solution.



CHAPTER 4

Systems of ODEs. Phase Plane. Qualitative Methods

Systems of ODEs have various applications (see, for instance, Secs. 4.1 and 4.5). Their theory is outlined in Sec. 4.2 and includes that of a single ODE. The practically important conversion of a single nth-order ODE to a system is shown in Sec. 4.1.

Linear systems (Secs. 4.3, 4.4, 4.6) are best treated by the use of vectors and matrices, of which, however, only a few elementary facts will be needed here, as given in Sec. 4.0 and probably familiar to most students.

Qualitative methods. In addition to actually solving systems (Sec. 4.3, 4.6), which is often difficult or even impossible, we shall explain a totally different method, namely, the powerful method of investigating the general behavior of whole families of solutions in the **phase plane** (Sec. 4.3). This approach to systems of ODEs is called a **qualitative method** because it does not need actual solutions (in contrats to a *"quantitative method"* of actually solving a system).

This *phase plane method,* as it is called, also gives information on **stability** of solutions, which is of general importance in control theory, circuit theory, population dynamics, and so on. Here, *stability of a physical system* means that, roughly speaking, a small change at some instant causes only small changes in the behavior of the system at all later times.

Phase plane methods can be extended to nonlinear systems, for which they are particularly useful. We will show this in Sec. 4.5, which includes a discussion of the pendulum equation and the Lotka-Volterra population model. We finally discuss nonhomogeneous linear systems in Sec. 4.6.

NOTATION. Analogous to Chaps. 1–3, we continue to denote unknown functions by y; thus, $y_1(t)$, $y_2(t)$. This seems preferable to suddenly using x for functions, $x_1(t)$, $x_2(t)$, as is sometimes done in systems of ODEs.

Prerequisite: Chap. 2.
References and Answers to Problems: App. 1 Part A, and App. 2.

4.0 Basics of Matrices and Vectors

In discussing *linear* systems of ODEs we shall use matrices and vectors. This simplifies formulas and clarifies ideas. But we shall need only a few elementary facts (by no means the bulk of material in Chaps. 7 and 8). These facts will very likely be at the disposal of most students. Hence ***this section is for reference only. Begin with Sec. 4.1 and consult 4.0 as needed.***

Most of our linear systems will consist of two ODEs in two unknown functions $y_1(t)$, $y_2(t)$,

$$(1)\qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2, \\ y_2' &= a_{21}y_1 + a_{22}y_2, \end{aligned} \qquad \text{for example,} \qquad \begin{aligned} y_1' &= -5y_1 + 2y_2 \\ y_2' &= 13y_1 + \tfrac{1}{2}y_2 \end{aligned}$$

(perhaps with additional *given* functions $g_1(t)$, $g_2(t)$ in the two ODEs on the right).

Similarly, a linear system of n first-order ODEs in n unknown functions $y_1(t), \cdots, y_n(t)$ is of the form

$$(2)\qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

(perhaps with an additional given function in each ODE on the right).

Some Definitions and Terms

Matrices. In (1) the (constant or variable) coefficients form a **2 × 2 matrix A,** that is, an array

$$(3)\qquad \mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \text{for example,} \qquad \mathbf{A} = \begin{bmatrix} -5 & 2 \\ 13 & \frac{1}{2} \end{bmatrix}.$$

Similarly, the coefficients in (2) form an **$n \times n$ matrix**

$$(4)\qquad \mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The a_{11}, a_{12}, $\cdots$ are called **entries,** the horizontal lines **rows,** and the vertical lines **columns.** Thus, in (3) the first row is $[a_{11} \quad a_{12}]$, the second row is $[a_{21} \quad a_{22}]$, and the first and second columns are

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

In the ***"double subscript notation"*** for entries, the first subscript denotes the *row* and the second the *column* in which the entry stands. Similarly in (4). The **main diagonal** is the diagonal $a_{11} \quad a_{22} \quad \cdots \quad a_{nn}$ in (4), hence $a_{11} \quad a_{22}$ in (3).

We shall need only **square matrices,** that is, matrices with the same number of rows and columns, as in (3) and (4).

Vectors. A **column vector x** with n **components** $x_1, \cdots, x_n$ is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{thus if } n = 2, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Similarly, a **row vector v** is of the form

$$\mathbf{v} = [v_1 \quad \cdots \quad v_n], \qquad \text{thus if } n = 2, \text{ then} \qquad \mathbf{v} = [v_1, \quad v_2].$$

Calculations with Matrices and Vectors

Equality. Two $n \times n$ matrices are *equal* if and only if corresponding entries are equal. Thus for $n = 2$, let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then $\mathbf{A} = \mathbf{B}$ if and only if

$$a_{11} = b_{11}, \qquad a_{12} = b_{12}$$
$$a_{21} = b_{21}, \qquad a_{22} = b_{22}.$$

Two column vectors (or two row vectors) are *equal* if and only if they both have n components and corresponding components are equal. Thus, let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad \text{Then} \qquad \mathbf{v} = \mathbf{x} \quad \text{if and only if} \quad \begin{matrix} v_1 = x_1 \\ v_2 = x_2. \end{matrix}$$

Addition is performed by adding corresponding entries (or components); here, matrices must both be $n \times n$, and vectors must both have the same number of components. Thus for $n = 2$,

$$\text{(5)} \qquad \mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}, \qquad \mathbf{v} + \mathbf{x} = \begin{bmatrix} v_1 + x_1 \\ v_2 + x_2 \end{bmatrix}.$$

Scalar multiplication (multiplication by a number c) is performed by multiplying each entry (or component) by c. For example, if

$$\mathbf{A} = \begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}, \qquad \text{then} \qquad -7\mathbf{A} = \begin{bmatrix} -63 & -21 \\ 14 & 0 \end{bmatrix}.$$

If

$$\mathbf{v} = \begin{bmatrix} 0.4 \\ -13 \end{bmatrix}, \qquad \text{then} \qquad 10\mathbf{v} = \begin{bmatrix} 4 \\ -130 \end{bmatrix}.$$

Matrix Multiplication. The product $\mathbf{C} = \mathbf{AB}$ (in this order) of two $n \times n$ matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ is the $n \times n$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$\text{(6)} \qquad c_{jk} = \sum_{m=1}^{n} a_{jm} b_{mk} \qquad \begin{array}{l} j = 1, \cdots, n \\ k = 1, \cdots, n, \end{array}$$

that is, multiply each entry in the jth *row* of $\mathbf{A}$ by the corresponding entry in the kth *column* of $\mathbf{B}$ and then add these n products. One says briefly that this is a "multiplication of rows into columns." For example,

$$\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9 \cdot 1 + 3 \cdot 2 & 9 \cdot (-4) + 3 \cdot 5 \\ -2 \cdot 1 + 0 \cdot 2 & (-2) \cdot (-4) + 0 \cdot 5 \end{bmatrix} = \begin{bmatrix} 15 & -21 \\ -2 & 8 \end{bmatrix}.$$

CAUTION! Matrix multiplication is ***not commutative,*** $\mathbf{AB} \neq \mathbf{BA}$ in general. In our example,

$$\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 9 + (-4) \cdot (-2) & 1 \cdot 3 + (-4) \cdot 0 \\ 2 \cdot 9 + 5 \cdot (-2) & 2 \cdot 3 + 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} 17 & 3 \\ 8 & 6 \end{bmatrix}.$$

Multiplication of an $n \times n$ matrix $\mathbf{A}$ by a vector $\mathbf{x}$ with n components is defined by the same rule: $\mathbf{v} = \mathbf{Ax}$ is the vector with the n components

$$v_j = \sum_{m=1}^{n} a_{jm} x_m \qquad j = 1, \cdots, n.$$

For example,

$$\begin{bmatrix} 12 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12x_1 + 7x_2 \\ -8x_1 + 3x_2 \end{bmatrix}.$$

Systems of ODEs as Vector Equations

Differentiation. The *derivative* of a matrix (or vector) with variable entries (or components) is obtained by differentiating each entry (or component). Thus, if

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ \sin t \end{bmatrix}, \quad \text{then} \quad \mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}.$$

Using matrix multiplication and differentiation, we can now write (1) as

$$\text{(7)} \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \mathbf{Ay} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{e.g.,} \quad \mathbf{y}' = \begin{bmatrix} -5 & 2 \\ 13 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Similarly for (2) by means of an $n \times n$ matrix $\mathbf{A}$ and a column vector $\mathbf{y}$ with n components, namely, $\mathbf{y}' = \mathbf{A}\mathbf{y}$. The vector equation (7) is equivalent to two equations for the components, and these are precisely the two ODEs in (1).

Some Further Operations and Terms

Transposition is the operation of writing columns as rows and conversely and is indicated by T. Thus the transpose $\mathbf{A}^\mathsf{T}$ of the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 13 & \frac{1}{2} \end{bmatrix} \quad \text{is} \quad \mathbf{A}^\mathsf{T} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 2 & \frac{1}{2} \end{bmatrix}.$$

The transpose of a column vector, say,

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad \text{is a row vector,} \qquad \mathbf{v}^\mathsf{T} = [v_1 \quad v_2],$$

and conversely.

Inverse of a Matrix. The $n \times n$ **unit matrix I** is the $n \times n$ matrix with main diagonal $1, 1, \cdots, 1$ and all other entries zero. If for a given $n \times n$ matrix $\mathbf{A}$ there is an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{A}$ is called **nonsingular** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$; thus

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{8}$$

If $\mathbf{A}$ has no inverse, it is called **singular.** For $n = 2$,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \tag{9}$$

where the **determinant** of $\mathbf{A}$ is

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{10}$$

(For general n, see Sec. 7.7, but this will not be needed in this chapter.)

Linear Independence. r given vectors $\mathbf{v}^{(1)}, \cdots, \mathbf{v}^{(r)}$ with n components are called a *linearly independent set* or, more briefly, **linearly independent,** if

$$c_1\mathbf{v}^{(1)} + \cdots + c_r\mathbf{v}^{(r)} = \mathbf{0} \tag{11}$$

implies that all scalars $c_1, \cdots, c_r$ must be zero; here, $\mathbf{0}$ denotes the **zero vector,** whose n components are all zero. If (11) also holds for scalars not all zero (so that at least one of these scalars is not zero), then these vectors are called a *linearly dependent set* or, briefly, **linearly dependent,** because then at least one of them can be expressed as a **linear combination** of the others; that is, if, for instance, $c_1 \neq 0$ in (11), then we can obtain

$$\mathbf{v}^{(1)} = -\frac{1}{c_1}(c_2\mathbf{v}^{(2)} + \cdots + c_r\mathbf{v}^{(r)}).$$

Eigenvalues, Eigenvectors

Eigenvalues and eigenvectors will be very important in this chapter (and, as a matter of fact, throughout mathematics).

Let $\mathbf{A} = [a_{jk}]$ be an $n \times n$ matrix. Consider the equation

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{12}$$

where λ is a scalar (a real or complex number) to be determined and $\mathbf{x}$ is a vector to be determined. Now for every λ a solution is $\mathbf{x} = \mathbf{0}$. A scalar λ such that (12) holds for some vector $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvalue** of **A**, and this vector is called an **eigenvector** of **A** corresponding to this eigenvalue λ.

We can write (12) as $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$ or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{13}$$

These are n linear algebraic equations in the n unknowns $x_1, \cdots, x_n$ (the components of $\mathbf{x}$). For these equations to have a solution $\mathbf{x} \neq \mathbf{0}$, the determinant of the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ must be zero. This is proved as a basic fact in linear algebra (Theorem 4 in Sec. 7.7). In this chapter we need this only for $n = 2$. Then (13) is

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \tag{14}$$

in components,

$$\begin{aligned} (a_{11} - \lambda)x_1 + \quad a_{12}x_2 \quad &= 0 \\ a_{21}x_1 \quad + (a_{22} - \lambda)x_2 &= 0. \end{aligned} \tag{14*}$$

Now $\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if its determinant $\det(\mathbf{A} - \lambda\mathbf{I})$, called the **characteristic determinant** of **A** (also for general n), is zero. This gives

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \end{aligned} \tag{15}$$

This quadratic equation in λ is called the **characteristic equation** of **A.** Its solutions are the eigenvalues λ_1 and λ_2 of **A.** First determine these. Then use (14*) with $\lambda = \lambda_1$ to determine an eigenvector $\mathbf{x}^{(1)}$ of **A** corresponding to λ_1. Finally use (14*) with $\lambda = \lambda_2$ to find an eigenvector $\mathbf{x}^{(2)}$ of **A** corresponding to λ_2. Note that if $\mathbf{x}$ is an eigenvector of **A,** so is $k\mathbf{x}$ for any $k \neq 0$.

EXAMPLE 1 Eigenvalue Problem

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}. \tag{16}$$

Solution. The characteristic equation is the quadratic equation

$$\det[\mathbf{A} - \lambda\mathbf{I}] = \begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0.$$

It has the solutions $\lambda_1 = -2$ and $\lambda_2 = -0.8$. These are the eigenvalues of **A**.

Eigenvectors are obtained from (14*). For $\lambda = \lambda_1 = -2$ we have from (14*)

$$\begin{aligned} (-4.0 + 2.0)x_1 + \quad 4.0x_2 \quad &= 0 \\ -1.6x_1 \quad + (1.2 + 2.0)x_2 &= 0. \end{aligned}$$

A solution of the first equation is $x_1 = 2$, $x_2 = 1$. This also satisfies the second equation. (Why?). Hence an eigenvector of **A** corresponding to $\lambda_1 = -2.0$ is

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \qquad \text{Similarly,} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \tag{17}$$

is an eigenvector of **A** corresponding to $\lambda_2 = -0.8$, as obtained from (14*) with $\lambda = \lambda_2$. Verify this. ■

4.1 Systems of ODEs as Models

We first illustrate with a few typical examples that systems of ODEs can serve as models in various applications. We further show that a higher order ODE (with the highest derivative standing alone on one side) can be reduced to a first-order system. Both facts account for the practical importance of these systems.

EXAMPLE 1 Mixing Problem Involving Two Tanks

A mixing problem involving a single tank is modeled by a single ODE, and you may first review the corresponding Example 3 in Sec. 1.3 because the principle of modeling will be the same for two tanks. The model will be a system of two first-order ODEs.

Tank T_1 and T_2 in Fig. 77 contain initially 100 gal of water each. In T_1 the water is pure, whereas 150 lb of fertilizer are dissolved in T_2. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?

Solution. *Step 1.* **Setting up the model.** As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. From Fig. 77 we see that

$$y_1' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_2 - \frac{2}{100}y_1 \qquad (\text{Tank } T_1)$$

$$y_2' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_1 - \frac{2}{100}y_2 \qquad (\text{Tank } T_2).$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs

$$\begin{aligned} y_1' &= -0.02y_1 + 0.02y_2 \qquad &(\text{Tank } T_1) \\ y_2' &= \ \ 0.02y_1 - 0.02y_2 \qquad &(\text{Tank } T_2). \end{aligned}$$

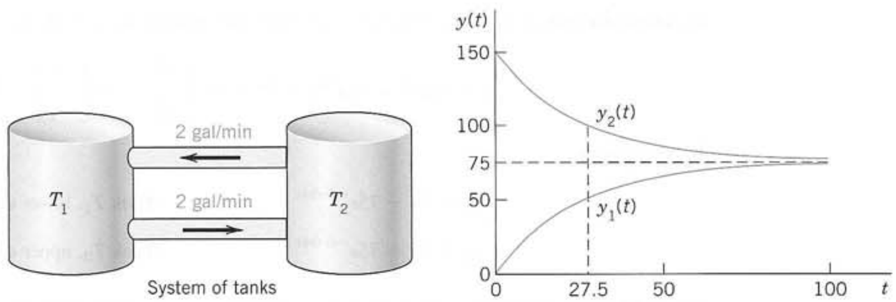


Fig. 77. Fertilizer content in Tanks T_1 (lower curve) and T_2

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix $\mathbf{A}$ this becomes

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

Step 2. **General solution.** As for a single equation, we try an exponential function of t,

$$\text{(1)} \qquad \mathbf{y} = \mathbf{x}e^{\lambda t}. \qquad \text{Then} \qquad \mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}.$$

Dividing the last equation $\lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

We need nontrivial solutions (solutions that are not identically zero). Hence we have to look for eigenvalues and eigenvectors of $\mathbf{A}$. The eigenvalues are the solutions of the characteristic equation

$$\text{(2)} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = (-0.02 - \lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0.$$

We see that $\lambda_1 = 0$ (which can very well happen—don't get mixed up—it is eigen*vectors* that must not be zero) and $\lambda_2 = -0.04$. Eigenvectors are obtained from (14*) in Sec. 4.0 with $\lambda = 0$ and $\lambda = -0.04$. For our present $\mathbf{A}$ this gives [we need only the first equation in (14*)]

$$-0.02x_1 + 0.02x_2 = 0 \qquad \text{and} \qquad (-0.02 + 0.04)x_1 + 0.02x_2 = 0,$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

From (1) and the superposition principle (which continues to hold for systems of homogeneous linear ODEs) we thus obtain a solution

$$\text{(3)} \qquad \mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

where c_1 and c_2 are arbitrary constants. Later we shall call this a **general solution.**

Step 3. **Use of initial conditions.** The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and (3) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)}e^{-0.04t} = 75\begin{bmatrix}1\\1\end{bmatrix} - 75\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}.$$

In components,

$$y_1 = 75 - 75e^{-0.04t} \qquad \text{(Tank } T_1\text{, lower curve)}$$

$$y_2 = 75 + 75e^{-0.04t} \qquad \text{(Tank } T_2\text{, upper curve).}$$

Figure 77 shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 lb. Did you expect this for physical reasons? Can you physically explain why the curves look "symmetric"? Would the limit change if T_1 initially contained 100 lb of fertilizer and T_2 contained 50 lb?

Step 4. **Answer.** T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 lb. Thus

$$y_1 = 75 - 75e^{-0.04t} = 50, \qquad e^{-0.04t} = \tfrac{1}{3}, \qquad t = (\ln 3)/0.04 = 27.5.$$

Hence the fluid should circulate for at least about half an hour. ■

EXAMPLE 2 Electrical Network

Find the currents $I_1(t)$ and $I_2(t)$ in the network in Fig. 78. Assume all currents and charges to be zero at $t = 0$, the instant when the switch is closed.

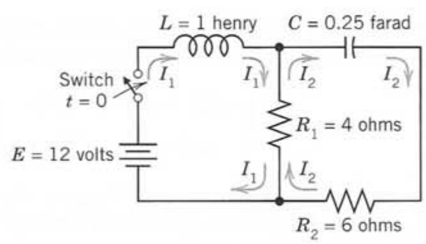


Fig. 78. Electrical network in Example 2

Solution. *Step 1.* **Setting up the mathematical model.** The model of this network is obtained from Kirchhoff's voltage law, as in Sec. 2.9 (where we considered single circuits). Let $I_1(t)$ and $I_2(t)$ be the currents in the left and right loops, respectively. In the left loop the voltage drops are $LI_1' = I_1'$ [V] over the inductor and $R_1(I_1 - I_2) = 4(I_1 - I_2)$ [V] over the resistor, the difference because I_1 and I_2 flow through the resistor in opposite directions. By Kirchhoff's voltage law the sum of these drops equals the voltage of the battery; that is, $I_1' + 4(I_1 - I_2) = 12$, hence

$$I_1' = -4I_1 + 4I_2 + 12. \tag{4a}$$

In the right loop the voltage drops are $R_2I_2 = 6I_2$ [V] and $R_1(I_2 - I_1) = 4(I_2 - I_1)$ [V] over the resistors and $(1/C)\int I_2\,dt = 4\int I_2\,dt$ [V] over the capacitor, and their sum is zero,

$$6I_2 + 4(I_2 - I_1) + 4\int I_2\,dt = 0 \qquad \text{or} \qquad 10I_2 - 4I_1 + 4\int I_2\,dt = 0.$$

Division by 10 and differentiation gives $I_2' - 0.4I_1' + 0.4I_2 = 0$.

To simplify the solution process, we first get rid of $0.4I_1'$, which by (4a) equals $0.4(-4I_1 + 4I_2 + 12)$. Substitution into the present ODE gives

$$I_2' = 0.4I_1' - 0.4I_2 = 0.4(-4I_1 + 4I_2 + 12) - 0.4I_2$$

and by simplification

$$I_2' = -1.6I_1 + 1.2I_2 + 4.8. \tag{4b}$$

In matrix form, (4) is (we write $\mathbf{J}$ since $\mathbf{I}$ is the unit matrix)

$$\mathbf{J}' = \mathbf{AJ} + \mathbf{g}, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}. \tag{5}$$

***Step 2.* Solving (5).** Because of the vector $\mathbf{g}$ this is a *nonhomogeneous* system, and we try to proceed as for a single ODE, solving first the *homogeneous* system $\mathbf{J}' = \mathbf{AJ}$ (thus $\mathbf{J}' - \mathbf{AJ} = \mathbf{0}$) by substituting $\mathbf{J} = \mathbf{x}e^{\lambda t}$. This gives

$$\mathbf{J}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}, \qquad \text{hence} \qquad \mathbf{Ax} = \lambda\mathbf{x}.$$

Hence to obtain a nontrivial solution, we again need the eigenvalues and eigenvectors. For the present matrix $\mathbf{A}$ they are derived in Example 1 in Sec. 4.0:

$$\lambda_1 = -2, \quad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -0.8, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}.$$

Hence a "general solution" of the homogeneous system is

$$\mathbf{J}_h = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t}.$$

For a particular solution of the nonhomogeneous system (5), since $\mathbf{g}$ is constant, we try a constant column vector $\mathbf{J}_p = \mathbf{a}$ with components a_1, a_2. Then $\mathbf{J}_p' = \mathbf{0}$, and substitution into (5) gives $\mathbf{Aa} + \mathbf{g} = \mathbf{0}$; in components,

$$\begin{aligned} -4.0a_1 + 4.0a_2 + 12.0 &= 0 \\ -1.6a_1 + 1.2a_2 + 4.8 &= 0. \end{aligned}$$

The solution is $a_1 = 3$, $a_2 = 0$; thus $\mathbf{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Hence

$$\mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a}; \tag{6}$$

in components,

$$\begin{aligned} I_1 &= 2c_1e^{-2t} + c_2e^{-0.8t} + 3 \\ I_2 &= c_1e^{-2t} + 0.8c_2e^{-0.8t}. \end{aligned}$$

The initial conditions give

$$\begin{aligned} I_1(0) &= 2c_1 + c_2 + 3 = 0 \\ I_2(0) &= c_1 + 0.8c_2 = 0. \end{aligned}$$

Hence $c_1 = -4$ and $c_2 = 5$. As the solution of our problem we thus obtain

$$\mathbf{J} = -4\mathbf{x}^{(1)}e^{-2t} + 5\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a}. \tag{7}$$

In components (Fig. 79b),

$$\begin{aligned} I_1 &= -8e^{-2t} + 5e^{-0.8t} + 3 \\ I_2 &= -4e^{-2t} + 4e^{-0.8t}. \end{aligned}$$

Now comes an important idea, on which we shall elaborate further, beginning in Sec. 4.3. Figure 79a shows $I_1(t)$ and $I_2(t)$ as two separate curves. Figure 79b shows these two currents as a single curve $[I_1(t), I_2(t)]$ in the I_1I_2-plane. This is a parametric representation with time t as the parameter. It is often important to know in which sense such a curve is traced. This can be indicated by an arrow in the sense of increasing t, as is shown. The I_1I_2-plane is called the **phase plane** of our system (5), and the curve in Fig. 79b is called a **trajectory.** We shall see that such **"phase plane representations"** are far more important than graphs as in Fig. 79a because they will give a much better qualitative overall impression of the general behavior of whole families of solutions, not merely of one solution as in the present case. ■

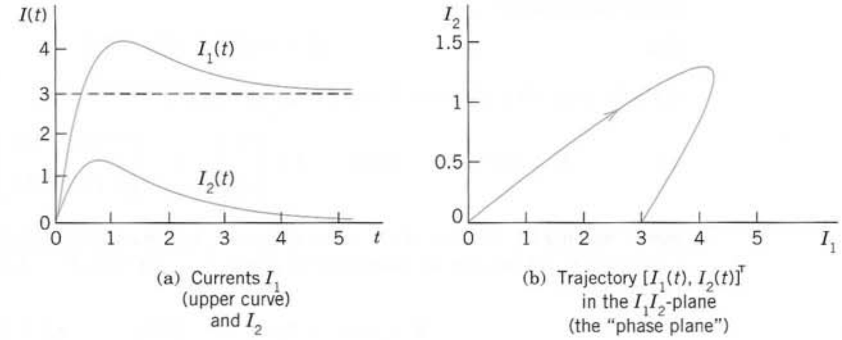


Fig. 79. Currents in Example 2

Conversion of an *n*th-Order ODE to a System

We show that an *n*th-order ODE of the general form (8) (see Theorem 1) can be converted to a system of n first-order ODEs. This is practically and theoretically important—practically because it permits the study and solution of single ODEs by methods for systems, and theoretically because it opens a way of including the theory of higher order ODEs into that of first-order systems. This conversion is another reason for the importance of systems, in addition to their use as models in various basic applications. The idea of the conversion is simple and straightforward, as follows.

THEOREM 1

Conversion of an ODE

An nth-order ODE

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{8}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}. \tag{9}$$

This system is of the form

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned} \tag{10}$$

PROOF The first $n - 1$ of these n ODEs follow immediately from (9) by differentiation. Also, $y_n' = y^{(n)}$ by (9), so that the last equation in (10) results from the given ODE (8). ■

EXAMPLE 3 Mass on a Spring

To gain confidence in the conversion method, let us apply it to an old friend of ours, modeling the free motions of a mass on a spring (see Sec. 2.4)

$$my'' + cy' + ky = 0 \qquad \text{or} \qquad y'' = -\frac{c}{m}y' - \frac{k}{m}y.$$

For this ODE (8) the system (10) is linear and homogeneous,

$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}\,y_1 - \frac{c}{m}\,y_2.$$

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

It agrees with that in Sec. 2.4. For an illustrative computation, let $m = 1$, $c = 2$, and $k = 0.75$. Then

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0.$$

This gives the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$. Eigenvectors follow from the first equation in $\mathbf{A} - \lambda\mathbf{I} = \mathbf{0}$, which is $-\lambda x_1 + x_2 = 0$. For λ_1 this gives $0.5x_1 + x_2 = 0$, say, $x_1 = 2$, $x_2 = -1$. For $\lambda_2 = -1.5$ it gives $1.5x_1 + x_2 = 0$, say, $x_1 = 1$, $x_2 = -1.5$. These eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \qquad \text{give} \qquad \mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

which is the expected solution. The second component is its derivative

$$y_2 = y_1' = y' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t}.$$

■

PROBLEM SET 4.1

1–6 MIXING PROBLEMS

1. Find out without calculation whether doubling the flow rate in Example 1 has the same effect as halfing the tank sizes. (Give a reason.)
2. What happens in Example 1 if we replace T_2 by a tank containing 500 gal of water and 150 lb of fertilizer dissolved in it?
3. Derive the eigenvectors in Example 1 without consulting this book.
4. In Example 1 find a "general solution" for any ratio a = (*flow rate*)/(*tank size*), tank sizes being equal. Comment on the result.
5. If you extend Example 1 by a tank T_3 of the same size as the others and connected to T_2 by two tubes with

flow rates as between T_1 and T_2, what system of ODEs will you get?

6. Find a "general solution" of the system in Prob. 5.

7–10 ELECTRICAL NETWORKS

7. Find the currents in Example 2 if the initial currents are 0 and -3 A (minus meaning that $I_2(0)$ flows against the direction of the arrow).

8. Find the currents in Example 2 if the resistance of R_1 and R_2 is doubled (general solution only). First, guess.

9. What are the limits of the currents in Example 2? Explain them in terms of physics.

10. Find the currents in Example 2 if the capacitance is changed to $C = 1/5.4$ F (farad).

11–15 CONVERSION TO SYSTEMS

Find a general solution of the given ODE (a) by first converting it to a system, (b), as given. (Show the details of your work.)

11. $y'' - 4y = 0$

12. $y'' + 2y' - 24y = 0$

13. $y'' - y' = 0$

14. $y'' + 15y' + 50y = 0$

15. $64y'' - 48y' - 7y = 0$

16. TEAM PROJECT. Two Masses on Springs. (a) Set up the model for the (undamped) system in Fig. 80.

(b) Solve the system of ODEs obtained. *Hint.* Try $\mathbf{y} = \mathbf{x}e^{\omega t}$ and set $\omega^2 = \lambda$. Proceed as in Example 1 or 2.

(c) Describe the influence of initial conditions on the possible kind of motions.

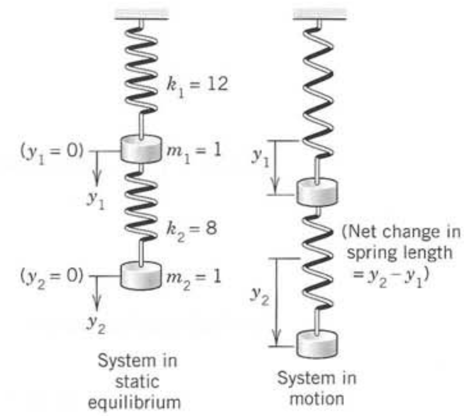


Fig. 80. Mechanical system in Team Project 16

4.2 Basic Theory of Systems of ODEs

In this section we discuss some basic concepts and facts about systems of ODEs that are quite similar to those for single ODEs.

The first-order systems in the last section were special cases of the more general system

$$
\begin{aligned}
y_1' &= f_1(t, y_1, \cdots, y_n) \\
y_2' &= f_2(t, y_1, \cdots, y_n) \\
&\cdots \\
y_n' &= f_n(t, y_1, \cdots, y_n).
\end{aligned} \tag{1}
$$

We can write the system (1) as a vector equation by introducing the column vectors $\mathbf{y} = [y_1 \ \cdots \ y_n]^\mathsf{T}$ and $\mathbf{f} = [f_1 \ \cdots \ f_n]^\mathsf{T}$ (where T means *transposition* and saves us the space that would be needed for writing $\mathbf{y}$ and $\mathbf{f}$ as columns). This gives

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}). \tag{1}$$

This system (1) includes almost all cases of practical interest. For $n = 1$ it becomes $y_1' = f_1(t, y_1)$ or, simply, $y' = f(t, y)$, well known to us from Chap. 1.

A **solution** of (1) on some interval $a < t < b$ is a set of n differentiable functions

$$y_1 = h_1(t), \quad \cdots, \quad y_n = h_n(t)$$

on $a < t < b$ that satisfy (1) throughout this interval. In vector form, introducing the *"solution vector"* $\mathbf{h} = [h_1 \quad \cdots \quad h_n]^\mathsf{T}$ (a column vector!) we can write

$$\mathbf{y} = \mathbf{h}(t).$$

An **initial value problem** for (1) consists of (1) and n given **initial conditions**

$$(2) \qquad y_1(t_0) = K_1, \qquad y_2(t_0) = K_2, \qquad \cdots, \qquad y_n(t_0) = K_n,$$

in vector form, $\mathbf{y}(t_0) = \mathbf{K}$, where t_0 is a specified value of t in the interval considered and the components of $\mathbf{K} = [K_1 \quad \cdots \quad K_n]^\mathsf{T}$ are given numbers. Sufficient conditions for the existence and uniqueness of a solution of an initial value problem (1), (2) are stated in the following theorem, which extends the theorems in Sec. 1.7 for a single equation. (For a proof, see Ref. [A7].)

THEOREM 1

Existence and Uniqueness Theorem

Let $f_1, \cdots, f_n$ in (1) be continuous functions having continuous partial derivatives $\partial f_1/\partial y_1, \cdots, \partial f_1/\partial y_n, \cdots, \partial f_n/\partial y_n$ in some domain R of $ty_1y_2 \cdots y_n$-space containing the point $(t_0, K_1, \cdots, K_n)$. Then (1) has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying (2), and this solution is unique.

Linear Systems

Extending the notion of a *linear* ODE, we call (1) a **linear system** if it is linear in $y_1 \cdots, y_n$; that is, if it can be written

$$(3) \qquad \begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ &\;\vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t). \end{aligned}$$

In vector form, this becomes

$$(3) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$$

$$\text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.$$

This system is called **homogeneous** if $\mathbf{g} = \mathbf{0}$, so that it is

$$(4) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y}.$$

If $\mathbf{g} \neq \mathbf{0}$, then (3) is called **nonhomogeneous.** The system in Example 1 in the last section is homogeneous and in Example 2 nonhomogeneous. The system in Example 3 is homogeneous.

For a linear system (3) we have $\partial f_1/\partial y_1 = a_{11}(t), \cdots, \partial f_n/\partial y_n = a_{nn}(t)$ in Theorem 1. Hence for a linear system we simply obtain the following.

THEOREM 2

Existence and Uniqueness in the Linear Case

Let the a_{jk}'s and g_j's in (3) be continuous functions of t on an open interval $\alpha < t < \beta$ containing the point $t = t_0$. Then (3) has a solution $\mathbf{y}(t)$ on this interval satisfying (2), and this solution is unique.

As for a single homogeneous linear ODE we have

THEOREM 3

Superposition Principle or Linearity Principle

If $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the **homogeneous linear** *system (4) on some interval, so is any linear combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$.*

PROOF Differentiating and using (4), we obtain

$$\begin{aligned}\mathbf{y}' &= \left[c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}\right]' \\ &= c_1\mathbf{y}^{(1)\prime} + c_2\mathbf{y}^{(2)\prime} \\ &= c_1\mathbf{A}\mathbf{y}^{(1)} + c_2\mathbf{A}\mathbf{y}^{(2)} \\ &= \mathbf{A}(c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}) = \mathbf{A}\mathbf{y}.\end{aligned}$$ ■

The general theory of linear systems of ODEs is quite similar to that of a single linear ODE in Secs. 2.6 and 2.7. To see this, we explain the most basic concepts and facts. For proofs we refer to more advanced texts, such as [A7].

Basis. General Solution. Wronskian

By a **basis** or a **fundamental system** of solutions of the homogeneous system (4) on some interval J we mean a linearly independent set of n solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ of (4) on that interval. (We write J because we need $\mathbf{I}$ to denote the unit matrix.) We call a corresponding linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} \cdots + c_n\mathbf{y}^{(n)} \qquad (c_1, \cdots, c_n \text{ arbitrary}) \tag{5}$$

a **general solution** of (4) on J. It can be shown that if the $a_{jk}(t)$ in (4) are continuous on J, then (4) has a basis of solutions on J, hence a general solution, which includes every solution of (4) on J.

We can write n solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ of (4) on some interval J as columns of an $n \times n$ matrix

$$\mathbf{Y} = \left[\mathbf{y}^{(1)} \quad \cdots \quad \mathbf{y}^{(n)}\right]. \tag{6}$$

The determinant of $\mathbf{Y}$ is called the **Wronskian** of $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$, written

$$(7) \qquad W(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \cdot & \cdot & \cdots & \cdot \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

The columns are these solutions, each in terms of components. These solutions form a basis on J if and only if W is not zero at any t_1 in this interval. W either is identically zero or is nowhere zero in J. (This is similar to Secs. 2.6 and 3.1.)

If the solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ in (5) form a basis (a fundamental system), then (6) is often called a **fundamental matrix.** Introducing a column vector $\mathbf{c} = [c_1 \quad c_2 \quad \cdots \quad c_n]^\mathsf{T}$, we can now write (5) simply as

$$(8) \qquad \mathbf{y} = \mathbf{Y}\mathbf{c}.$$

Furthermore, we can relate (7) to Sec. 2.6, as follows. If y and z are solutions of a second-order homogeneous linear ODE, their Wronskian is

$$W(y, z) = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}.$$

To write this ODE as a system, we have to set $y = y_1$, $y' = y_1' = y_2$ and similarly for z (see Sec. 4.1). But then $W(y, z)$ becomes (7), except for notation.

4.3 Constant-Coefficient Systems. Phase Plane Method

Continuing, we now assume that our homogeneous linear system

$$(1) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y}$$

under discussion has ***constant coefficients,*** so that the $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ has entries not depending on t. We want to solve (1). Now a single ODE $y' = ky$ has the solution $y = Ce^{kt}$. So let us try

$$(2) \qquad \mathbf{y} = \mathbf{x}e^{\lambda t}.$$

Substitution into (1) gives $\mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t}$. Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**

$$(3) \qquad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Thus the nontrivial solutions of (1) (solutions that are not zero vectors) are of the form (2), where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is a corresponding eigenvector.

We assume that $\mathbf{A}$ has a linearly independent set of n eigenvectors. This holds in most applications, in particular if $\mathbf{A}$ is symmetric ($a_{kj} = a_{jk}$) or skew-symmetric ($a_{kj} = -a_{jk}$) or has n *different* eigenvalues.

Let those eigenvectors be $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ and let them correspond to eigenvalues $\lambda_1, \cdots, \lambda_n$ (which may be all different, or some—or even all—may be equal). Then the corresponding solutions (2) are

$$\mathbf{y}^{(1)} = \mathbf{x}^{(1)}e^{\lambda_1 t}, \quad \cdots, \quad \mathbf{y}^{(n)} = \mathbf{x}^{(n)}e^{\lambda_n t}. \tag{4}$$

Their Wronskian $W = W(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)})$ [(7) in Sec. 4.2] is given by

$$W = (\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}) = \begin{vmatrix} x_1^{(1)}e^{\lambda_1 t} & \cdots & x_1^{(n)}e^{\lambda_n t} \\ x_2^{(1)}e^{\lambda_1 t} & \cdots & x_2^{(n)}e^{\lambda_n t} \\ \cdot & \cdots & \cdot \\ x_n^{(1)}e^{\lambda_1 t} & \cdots & x_n^{(n)}e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t + \cdots + \lambda_n t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}.$$

On the right, the exponential function is never zero, and the determinant is not zero either because its columns are the n linearly independent eigenvectors. This proves the following theorem, whose assumption is true if the matrix $\mathbf{A}$ is symmetric or skew-symmetric, or if the n eigenvalues of $\mathbf{A}$ are all different.

THEOREM 1

General Solution

If the constant matrix $\mathbf{A}$ *in the system* (1) *has a linearly independent set of* n *eigenvectors, then the corresponding solutions* $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ *in* (4) *form a basis of solutions of* (1), *and the corresponding general solution is*

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + \cdots + c_n\mathbf{x}^{(n)}e^{\lambda_n t}. \tag{5}$$

How to Graph Solutions in the Phase Plane

We shall now concentrate on systems (1) with constant coefficients consisting of two ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y}; \qquad \text{in components,} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{6}$$

Of course, we can graph solutions of (6),

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{7}$$

as two curves over the t-axis, one for each component of $\mathbf{y}(t)$. (Figure 79a in Sec. 4.1 shows an example.) But we can also graph (7) as a single curve in the y_1y_2-plane. This is a *parametric representation* (*parametric equation*) with parameter t. (See Fig. 79b for an example. Many more follow. Parametric equations also occur in calculus.) Such a curve is called a **trajectory** (or sometimes an *orbit* or *path*) of (6). The y_1y_2-plane is called the **phase plane.**[1] If we fill the phase plane with trajectories of (6), we obtain the so-called **phase portrait** of (6).

EXAMPLE 1 Trajectories in the Phase Plane (Phase Portrait)

In order to see what is going on, let us find and graph solutions of the system

$$\text{(8)} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= y_1 - 3y_2. \end{aligned}$$

Solution. By substituting $\mathbf{y} = \mathbf{x}e^{\lambda t}$ and $\mathbf{y}' = \lambda\mathbf{x}e^{\lambda t}$ and dropping the exponential function we get $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0.$$

This gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$. Eigenvectors are then obtained from

$$(-3-\lambda)x_1 + x_2 = 0.$$

For $\lambda_1 = -2$ this is $-x_1 + x_2 = 0$. Hence we can take $\mathbf{x}^{(1)} = [1 \quad 1]^\mathsf{T}$. For $\lambda_2 = -4$ this becomes $x_1 + x_2 = 0$, and an eigenvector is $\mathbf{x}^{(2)} = [1 \quad -1]^\mathsf{T}$. This gives the general solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}.$$

Figure 81 on p. 142 shows a phase portrait of some of the trajectories (to which more trajectories could be added if so desired). The two straight trajectories correspond to $c_1 = 0$ and $c_2 = 0$ and the others to other choices of c_1, c_2. ■

Studies of solutions in the phase plane have recently become quite important, along with advances in computer graphics, because a phase portrait gives a good general qualitative impression of the entire family of solutions. This method becomes particularly valuable in the frequent cases when solving an ODE or a system is inconvenient or impossible.

Critical Points of the System (6)

The point $\mathbf{y} = \mathbf{0}$ in Fig. 81 seems to be a common point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus. Indeed, from (6) we obtain

$$\text{(9)} \qquad \frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}.$$

[1] A name that comes from physics, where it is the y-(mv)-plane, used to plot a motion in terms of position y and velocity $y' = v$ (m = mass); but the name is now used quite generally for the y_1y_2-plane.

The use of the phase plane is a **qualitative method,** a method of obtaining general qualitative information on solutions without actually solving an ODE or a system. This method was created by HENRI POINCARÉ (1854–1912), a great French mathematician, whose work was also fundamental in complex analysis, divergent series, topology, and astronomy.

This associates with every point P: (y_1, y_2) a unique tangent direction dy_2/dy_1 of the trajectory passing through P, except for the point $P = P_0$: $(0, 0)$, where the right side of (9) becomes 0/0. This point P_0, at which dy_2/dy_1 becomes undetermined, is called a **critical point** of (6).

Five Types of Critical Points

There are five types of critical points depending on the geometric shape of the trajectories near them. They are called **improper nodes, proper nodes, saddle points, centers,** and **spiral points.** We define and illustrate them in Examples 1–5.

EXAMPLE 1 ***(Continued)* Improper Node (Fig. 81)**

An **improper node** is a critical point P_0 at which all the trajectories, except for two of them, have the same limiting direction of the tangent. The two exceptional trajectories also have a limiting direction of the tangent at P_0 which, however, is different.

The system (8) has an improper node at **0**, as its phase portrait Fig. 81 shows. The common limiting direction at **0** is that of the eigenvector $\mathbf{x}^{(1)} = [1 \quad 1]^\mathsf{T}$ because e^{-4t} goes to zero faster than e^{-2t} as t increases. The two exceptional limiting tangent directions are those of $\mathbf{x}^{(2)} = [1 \quad -1]^\mathsf{T}$ and $-\mathbf{x}^{(2)} = [-1 \quad 1]^\mathsf{T}$. ■

EXAMPLE 2 **Proper Node (Fig. 82)**

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction **d** at P_0 there is a trajectory having **d** as its limiting direction.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= y_2 \end{aligned} \tag{10}$$

has a proper node at the origin (see Fig. 82). Indeed, the matrix is the unit matrix. Its characteristic equation $(1 - \lambda)^2 = 0$ has the root $\lambda = 1$. Any $\mathbf{x} \neq \mathbf{0}$ is an eigenvector, and we can take $[1 \quad 0]^\mathsf{T}$ and $[0 \quad 1]^\mathsf{T}$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^t \end{aligned} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1. \quad ■$$

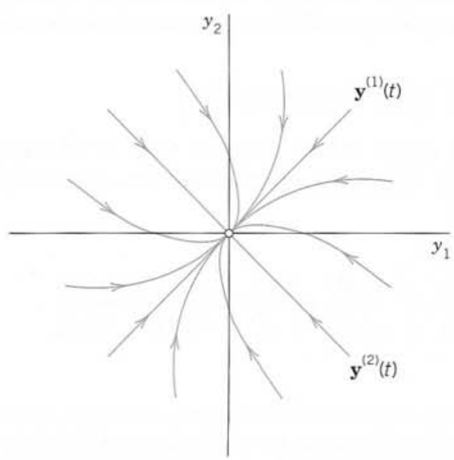


Fig. 81. Trajectories of the system (8) (Improper node)

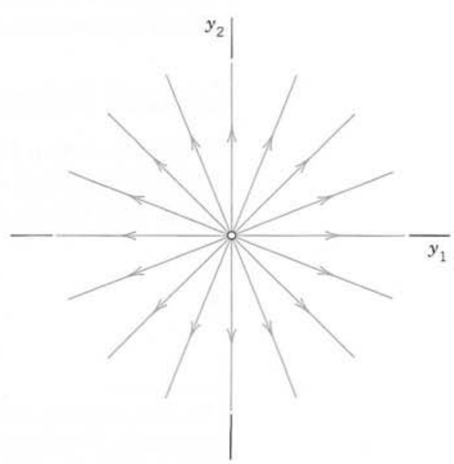


Fig. 82. Trajectories of the system (10) (Proper node)

EXAMPLE 3 Saddle Point (Fig. 83)

A **saddle point** is a critical point P_0 at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= -y_2 \end{aligned} \tag{11}$$

has a saddle point at the origin. Its characteristic equation $(1 - \lambda)(-1 - \lambda) = 0$ has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ an eigenvector $[1 \quad 0]^\mathsf{T}$ is obtained from the second row of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $0x_1 + (-1 - 1)x_2 = 0$. For $\lambda_2 = -1$ the first row gives $[0 \quad 1]^\mathsf{T}$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned} \qquad \text{or} \qquad y_1 y_2 = const.$$

This is a family of hyperbolas (and the coordinate axes); see Fig. 83. ■

EXAMPLE 4 Center (Fig. 84)

A **center** is a critical point that is enclosed by infinitely many closed trajectories.

The system

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 \end{aligned} \tag{12}$$

has a center at the origin. The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2i$ and $-2i$. For $2i$ an eigenvector follows from the first equation $-2ix_1 + x_2 = 0$ of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, say, $[1 \quad 2i]^\mathsf{T}$. For $\lambda = -2i$ that equation is $-(-2i)x_1 + x_2 = 0$ and gives, say, $[1 \quad -2i]^\mathsf{T}$. Hence a complex general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}, \qquad \text{thus} \qquad \begin{aligned} y_1 &= c_1 e^{2it} + c_2 e^{-2it} \\ y_2 &= 2ic_1 e^{2it} - 2ic_2 e^{-2it}. \end{aligned} \tag{12*}$$

The next step would be the transformation of this solution to real form by the Euler formula (Sec. 2.2). But we were just curious to see what kind of eigenvalues we obtain in the case of a center. Accordingly, we do not continue, but start again from the beginning and use a shortcut. We rewrite the given equations in the form $y_1' = y_2$, $4y_1 = -y_2'$; then the product of the left sides must equal the product of the right sides,

$$4y_1 y_1' = -y_2 y_2'. \qquad \text{By integration,} \qquad 2y_1^2 + \tfrac{1}{2}y_2^2 = const.$$

This is a family of ellipses (see Fig. 84) enclosing the center at the origin. ■

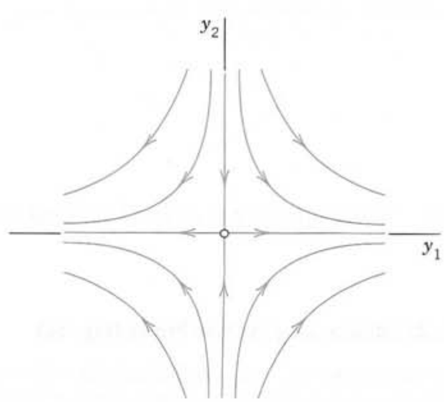


Fig. 83. Trajectories of the system (11) (Saddle point)

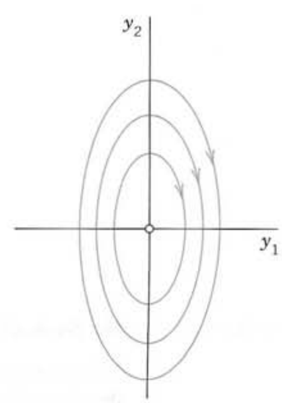


Fig. 84. Trajectories of the system (12) (Center)

EXAMPLE 5 Spiral Point (Fig. 85)

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$ (or tracing these spirals in the opposite sense, away from P_0).

The system

$$\text{(13)} \qquad \mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}$$

has a spiral point at the origin, as we shall see. The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. It gives the eigenvalues $-1 + i$ and $-1 - i$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For $\lambda = -1 + i$ this becomes $-ix_1 + x_2 = 0$ and we can take $[1 \quad i]^\mathsf{T}$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - i$ is $[1 \quad -i]^\mathsf{T}$. This gives the complex general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}.$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. But, as in the last example, we just wanted to see what eigenvalues to expect in the case of a spiral point. Accordingly, we start again from the beginning and instead of that rather lengthy systematic calculation we use a shortcut. We multiply the first equation in (13) by y_1, the second by y_2, and add, obtaining

$$y_1 y_1' + y_2 y_2' = -(y_1{}^2 + y_2{}^2).$$

We now introduce polar coordinates r, t, where $r^2 = y_1{}^2 + y_2{}^2$. Differentiating this with respect to t gives $2rr' = 2y_1 y_1' + 2y_2 y_2'$. Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real c this is a spiral, as claimed. (see Fig. 85). ■

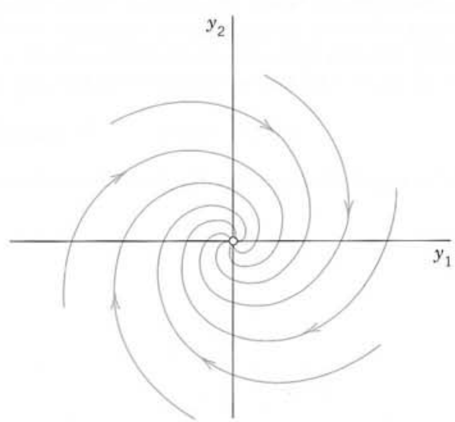


Fig. 85. Trajectories of the system (13) (Spiral point)

EXAMPLE 6 No Basis of Eigenvectors Available. Degenerate Node (Fig. 86)

This cannot happen if $\mathbf{A}$ in (1) is symmetric ($a_{kj} = a_{jk}$, as in Examples 1–3) or skew-symmetric ($a_{kj} = -a_{jk}$, thus $a_{jj} = 0$). And it does not happen in many other cases (see Examples 4 and 5). Hence it suffices to explain the method to be used by an example.

Find and graph a general solution of

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}. \tag{14}$$

Solution. $\mathbf{A}$ is not skew-symmetric! Its characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

It has a double root $\lambda = 3$. Hence eigenvectors are obtained from $(4 - \lambda)x_1 + x_2 = 0$, thus from $x_1 + x_2 = 0$, say, $\mathbf{x}^{(1)} = [1 \quad -1]^\mathsf{T}$ and nonzero multiples of it (which do not help). The method now is to substitute

$$\mathbf{y}^{(2)} = \mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

with constant $\mathbf{u} = [u_1 \quad u_2]^\mathsf{T}$ into (14). (The $\mathbf{x}t$-term alone, the analog of what we did in Sec. 2.2 in the case of a double root, would not be enough. Try it.) This gives

$$\mathbf{y}^{(2)\prime} = \mathbf{x}e^{\lambda t} + \lambda\mathbf{x}te^{\lambda t} + \lambda\mathbf{u}e^{\lambda t} = \mathbf{A}\mathbf{y}^{(2)} = \mathbf{A}\mathbf{x}te^{\lambda t} + \mathbf{A}\mathbf{u}e^{\lambda t}.$$

On the right, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Hence the terms $\lambda\mathbf{x}te^{\lambda t}$ cancel, and then division by $e^{\lambda t}$ gives

$$\mathbf{x} + \lambda\mathbf{u} = \mathbf{A}\mathbf{u}, \qquad \text{thus} \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{x}.$$

Here $\lambda = 3$ and $\mathbf{x} = [1 \quad -1]^\mathsf{T}$, so that

$$(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{thus} \qquad \begin{aligned} u_1 + u_2 &= 1 \\ -u_1 - u_2 &= -1. \end{aligned}$$

A solution, linearly independent of $\mathbf{x} = [1 \quad -1]^\mathsf{T}$, is $\mathbf{u} = [0 \quad 1]^\mathsf{T}$. This yields the answer (Fig. 86)

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}.$$

The critical point at the origin is often called a **degenerate node.** $c_1\mathbf{y}^{(1)}$ gives the heavy straight line, with $c_1 > 0$ the lower part and $c_1 < 0$ the upper part of it. $\mathbf{y}^{(2)}$ gives the right part of the heavy curve from 0 through the second, first, and—finally—fourth quadrants. $-\mathbf{y}^{(2)}$ gives the other part of that curve. ■

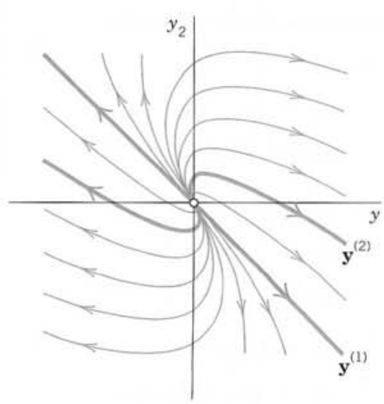


Fig. 86. Degenerate node in Example 6

We mention that for a system (1) with three or more equations and a triple eigenvalue with only one linearly independent eigenvector, one will get two solutions, as just discussed, and a third linearly independent one from

$$\mathbf{y}^{(3)} = \tfrac{1}{2}\mathbf{x}t^2e^{\lambda t} + \mathbf{u}te^{\lambda t} + \mathbf{v}e^{\lambda t} \qquad \text{with } \mathbf{v} \text{ from} \qquad \mathbf{u} + \lambda\mathbf{v} = \mathbf{A}\mathbf{v}.$$

PROBLEM SET 4.3

1–9 GENERAL SOLUTION

Find a real general solution of the following systems. (Show the details.)

1. $y_1' = 3y_2$, $y_2' = 12y_1$

2. $y_1' = 5y_2$, $y_2' = 5y_1$

3. $y_1' = y_1 + y_2$, $y_2' = y_1 + y_2$

4. $y_1' = 9y_1 + 13.5y_2$, $y_2' = 1.5y_1 + 9y_2$

5. $y_1' = 4y_2$, $y_2' = -4y_1$

6. $y_1' = 2y_1 - 2y_2$, $y_2' = 2y_1 + 2y_2$

7. $y_1' = 2y_1 + 8y_2 - 4y_3$, $y_2' = -4y_1 - 10y_2 + 2y_3$, $y_3' = -4y_1 - 4y_2 - 4y_3$

8. $y_1' = 8y_1 - y_2$, $y_2' = y_1 + 10y_2$

9. $y_1' = -y_1 + y_2 + 0.4y_3$, $y_2' = y_1 - 0.1y_2 + 1.4y_3$, $y_3' = 0.4y_1 + 1.4y_2 + 0.2y_3$

10–15 INITIAL VALUE PROBLEMS

Solve the following initial value problems. (Show the details.)

10. $y_1' = y_1 + y_2$, $y_2' = 4y_1 + y_2$, $y_1(0) = 1,\ y_2(0) = 6$

11. $y_1' = y_1 + 2y_2$, $y_2' = \tfrac{1}{2}y_1 + y_2$, $y_1(0) = 16,\ y_2(0) = -2$

12. $y_1' = 3y_1 + 2y_2$, $y_2' = 2y_1 + 3y_2$, $y_1(0) = 7,\ y_2(0) = 7$

13. $y_1' = \tfrac{1}{2}y_1 - 2y_2$, $y_2' = -\tfrac{3}{2}y_1 + y_2$, $y_1(0) = 0.4,\ y_2(0) = 3.8$

14. $y_1' = -y_1 + 5y_2$, $y_2' = -y_1 + 3y_2$, $y_1(0) = 7,\ y_2(0) = 2$

15. $y_1' = 2y_1 + 5y_2$, $y_2' = 5y_1 + 12.5y_2$, $y_1(0) = 12,\ y_2(0) = 1$

16–17 CONVERSION

Find a general solution by conversion to a single ODE.

16. The system in Prob. 8.

17. The system in Example 5.

18. (Mixing problem, Fig. 87) Each of the two tanks contains 400 gal of water, in which initially 100 lb (Tank T_1) and 40 lb (Tank T_2) of fertilizer are dissolved. The inflow, circulation, and outflow are shown in Fig. 87. The mixture is kept uniform by stirring. Find the fertilizer contents $y_1(t)$ in T_1 and $y_2(t)$ in T_2.

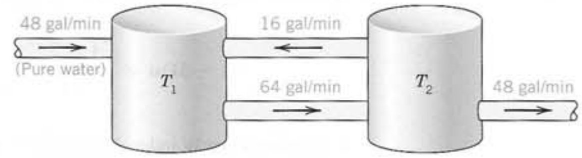


Fig. 87. Tanks in Problem 18

19. (Network) Show that a model for the currents $I_1(t)$ and $I_2(t)$ in Fig. 88 is

$$\frac{1}{C}\int I_1\,dt + R(I_1 - I_2) = 0, \qquad LI_2' + R(I_2 - I_1) = 0.$$

Find a general solution, assuming that $R = 20\ \Omega$, $L = 0.5$ H, $C = 2 \cdot 10^{-4}$ F.

20. CAS PROJECT. Phase Portraits. Graph some of the figures in this section, in particular Fig. 86 on the degenerate node, in which the vector $\mathbf{y}^{(2)}$ depends on t. In each figure highlight a trajectory that satisfies an initial condition of your choice.

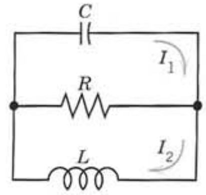


Fig. 88. Network in Problem 19

4.4 Criteria for Critical Points. Stability

We continue our discussion of homogeneous linear systems with constant coefficients

$$(1)\quad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}, \qquad \text{in components,} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

From the examples in the last section we have seen that we can obtain an overview of families of solution curves if we represent them parametrically as $\mathbf{y}(t) = [y_1(t) \quad y_2(t)]^\mathsf{T}$ and graph them as curves in the y_1y_2-plane, called the **phase plane.** Such a curve is called a **trajectory** of (1), and their totality is known as the **phase portrait** of (1).

Now we have seen that solutions are of the form

$$\mathbf{y}(t) = \mathbf{x}e^{\lambda t}. \qquad \text{Substitution into (1) gives} \qquad \mathbf{y}'(t) = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t}.$$

Dropping the common factor $e^{\lambda t}$, we have

$$(2) \qquad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Hence $\mathbf{y}(t)$ is a (nonzero) solution of (1) if λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ a corresponding eigenvector.

Our examples in the last section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system (1) defined as a point at which dy_2/dy_1 becomes undetermined, 0/0; here [see (9) in Sec. 4.3]

$$(3) \qquad \frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}.$$

We also recall from Sec. 4.3 that there are various types of critical points, and we shall now see how these types are related to the eigenvalues. The latter are solutions $\lambda = \lambda_1$ and λ_2 of the characteristic equation

$$(4) \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A} = 0.$$

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with coefficients p, q and discriminant Δ given by

$$(5) \qquad p = a_{11} + a_{22}, \qquad q = \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta = p^2 - 4q.$$

From calculus we know that the solutions of this equation are

$$(6) \qquad \lambda_1 = \tfrac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \tfrac{1}{2}(p - \sqrt{\Delta}).$$

Furthermore, the product representation of the equation gives

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Hence p is the sum and q the product of the eigenvalues. Also $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ from (6). Together,

$$p = \lambda_1 + \lambda_2, \qquad q = \lambda_1 \lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2. \tag{7}$$

This gives the criteria in Table 4.1 for classifying critical points. A derivation will be indicated later in this section.

Table 4.1 Eigenvalue Criteria for Critical Points (Derivation after Table 4.2)

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q > 0$	$\Delta \geqq 0$	Real, same sign
(b) Saddle point		$q < 0$		Real, opposite sign
(c) Center	$p = 0$	$q > 0$		Pure imaginary
(d) Spiral point	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

Stability

Critical points may also be classified in terms of their stability. Stability concepts are basic in engineering and other applications. They are suggested by physics, where **stability** means, roughly speaking, that a small change (a small disturbance) of a physical system at some instant changes the behavior of the system only slightly at all future times t. For critical points, the following concepts are appropriate.

DEFINITIONS

Stable, Unstable, Stable and Attractive

A critical point P_0 of (1) is called **stable**[2] if, roughly, all trajectories of (1) that at some instant are close to P_0 remain close to P_0 at all future times; precisely: if for every disk D_ϵ of radius $\epsilon > 0$ with center P_0 there is a disk D_δ of radius $\delta > 0$ with center P_0 such that every trajectory of (1) that has a point P_1 (corresponding to $t = t_1$, say) in D_δ has all its points corresponding to $t \geqq t_1$ in D_ϵ. See Fig. 89.

P_0 is called **unstable** if P_0 is not stable.

P_0 is called **stable and attractive** (or *asymptotically stable*) if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \to \infty$. See Fig. 90.

Classification criteria for critical points in terms of stability are given in Table 4.2. Both tables are summarized in the **stability chart** in Fig. 91. In this chart the region of instability is dark blue.

[2]In the sense of the Russian mathematician ALEXANDER MICHAILOVICH LJAPUNOV (1857–1918), whose work was fundamental in stability theory for ODEs. This is perhaps the most appropriate definition of stability (and the only we shall use), but there are others, too.

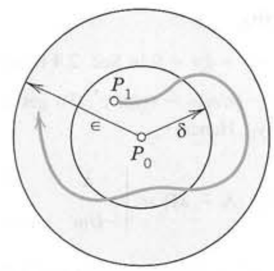


Fig. 89. Stable critical point P_0 of (1) (The trajectory initiating at P_1 stays in the disk of radius ϵ.)

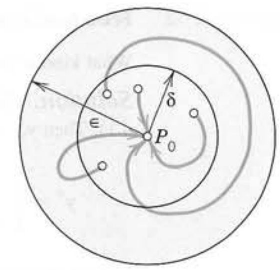


Fig. 90. Stable and attractive critical point P_0 of (1)

Table 4.2 **Stability Criteria for Critical Points**

Type of Stability	$p = \lambda_1 + \lambda_2$		$q = \lambda_1\lambda_2$
(a) Stable and attractive	$p < 0$		$q > 0$
(b) Stable	$p \leqq 0$		$q > 0$
(c) Unstable	$p > 0$	OR	$q < 0$

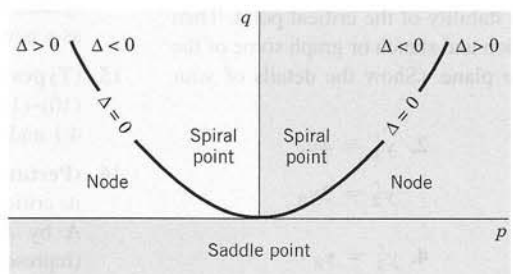


Fig. 91. Stability chart of the system (1) with p, q, Δ defined in (5). Stable and attractive: The second quadrant without the q-axis. Stability also on the positive q-axis (which corresponds to centers). Unstable: Dark blue region

We indicate how the criteria in Tables 4.1 and 4.2 are obtained. If $q = \lambda_1\lambda_2 > 0$, both eigenvalues are positive or both are negative or complex conjugates. If also $p = \lambda_1 + \lambda_2 < 0$, both are negative or have a negative real part. Hence P_0 is stable and attractive. The reasoning for the other two lines in Table 4.2 is similar.

If $\Delta < 0$, the eigenvalues are complex conjugates, say, $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. If also $p = \lambda_1 + \lambda_2 = 2\alpha < 0$, this gives a spiral point that is stable and attractive. If $p = 2\alpha > 0$, this gives an unstable spiral point.

If $p = 0$, then $\lambda_2 = -\lambda_1$ and $q = \lambda_1\lambda_2 = -\lambda_1{}^2$. If also $q > 0$, then $\lambda_1{}^2 = -q < 0$, so that λ_1, and thus λ_2, must be pure imaginary. This gives periodic solutions, their trajectories being closed curves around P_0, which is a center.

EXAMPLE 1 Application of the Criteria in Tables 4.1 and 4.2

In Example 1, Sec. 4.3, we have $\mathbf{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$, $p = -6$, $q = 8$, $\Delta = 4$, a node by Table 4.1(a), which is stable and attractive by Table 4.2(a). ■

EXAMPLE 2 Free Motions of a Mass on a Spring

What kind of critical point does $my'' + cy' + ky = 0$ in Sec. 2.4 have?

Solution. Division by m gives $y'' = -(k/m)y - (c/m)y'$. To get a system, set $y_1 = y$, $y_2 = y'$ (see Sec. 4.1). Then $y_2' = y'' = -(k/m)y_1 - (c/m)y_2$. Hence

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \mathbf{y}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

We see that $p = -c/m$, $q = k/m$, $\Delta = (c/m)^2 - 4k/m$. From this and Tables 4.1 and 4.2 we obtain the following results. Note that in the last three cases the discriminant Δ plays an essential role.

No damping. $c = 0$, $p = 0$, $q > 0$, a center.
Underdamping. $c^2 < 4mk$, $p < 0$, $q > 0$, $\Delta < 0$, a stable and attractive spiral point.
Critical damping. $c^2 = 4mk$, $p < 0$, $q > 0$, $\Delta = 0$, a stable and attractive node.
Overdamping. $c^2 > 4mk$, $p < 0$, $q > 0$, $\Delta > 0$, a stable and attractive node. ■

PROBLEM SET 4.4

1–9 TYPE AND STABILITY OF CRITICAL POINT

Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane. (Show the details of your work.)

1. $y_1' = 2y_2$, $y_2' = 8y_1$

2. $y_1' = 4y_1$, $y_2' = 3y_2$

3. $y_1' = 2y_1 + y_2$, $y_2' = y_1 + 2y_2$

4. $y_1' = y_2$, $y_2' = -5y_1 - 2y_2$

5. $y_1' = -4y_1 + y_2$, $y_2' = y_1 - 4y_2$

6. $y_1' = y_1 + 10y_2$, $y_2' = 7y_1 - 8y_2$

7. $y_1' = -2y_2$, $y_2' = 8y_1$

8. $y_1' = 3y_1 + 5y_2$, $y_2' = -5y_1 - 3y_2$

9. $y_1' = y_1 + 2y_2$, $y_2' = 2y_1 + y_2$

10–12 FORM OF TRAJECTORIES

What kind of curves are the trajectories of the following ODEs in the phase plane?

10. $y'' + 5y' = 0$

11. $y'' - k^2y = 0$

12. $y'' + \frac{1}{16}y = 0$

13. (Damped oscillation) Solve $y'' + 4y' + 5y = 0$. What kind of curves do you get as trajectories?

14. (Transformation of variable) What happens to the system (1) and its critical point if you introduce $\tau = -t$ as a new independent variable?

15. (Types of critical points) Discuss the critical points in (10)–(14) in Sec. 4.3 by applying the criteria in Tables 4.1 and 4.2 in this section.

16. (Perturbation of center) If a system has a center as its critical point, what happens if you replace the matrix $\mathbf{A}$ by $\widetilde{\mathbf{A}} = \mathbf{A} + k\mathbf{I}$ with any real number $k \neq 0$ (representing measurement errors in the diagonal entries)?

17. (Perturbation) The system in Example 4 in Sec. 4.3 has a center as its critical point. Replace each a_{jk} in Example 4, Sec. 4.3, by $a_{jk} + b$. Find values of b such that you get (a) a saddle point, (b) a stable and attractive node, (c) a stable and attractive spiral, (d) an unstable spiral, (e) an unstable node.

18. CAS EXPERIMENT. Phase Portraits. Graph phase portraits for the systems in Prob. 17 with the values of b suggested in the answer. Try to illustrate how the phase portrait changes "continuously" under a continuous change of b.

19. WRITING EXPERIMENT. Stability. Stability concepts are basic in physics and engineering. Write a two-part report of 3 pages each (A) on general applications in which stability plays a role (be as precise as you can), and (B) on material related to stability in this section. Use your own formulations and examples; do not copy.

20. (Stability chart) Locate the critical points of the systems (10)–(14) in Sec. 4.3 and of Probs. 1, 3, 5 in this problem set on the stability chart.

4.5 Qualitative Methods for Nonlinear Systems

Qualitative methods are methods of obtaining qualitative information on solutions without actually solving a system. These methods are particularly valuable for systems whose solution by analytic methods is difficult or impossible. This is the case for many practically important **nonlinear systems**

$$(1) \qquad \mathbf{y}' = \mathbf{f}(\mathbf{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned}$$

In this section we extend **phase plane methods,** as just discussed, from linear systems to nonlinear systems (1). We assume that (1) is **autonomous,** that is, the independent variable t does not occur explicitly. (All examples in the last section are autonomous.) We shall again exhibit entire families of solutions. This is an advantage over numeric methods, which give only one (approximate) solution at a time.

Concepts needed from the last section are the **phase plane** (the y_1y_2-plane), **trajectories** (solution curves of (1) in the phase plane), the **phase portrait** of (1) (the totality of these trajectories), and **critical points** of (1) (points (y_1, y_2) at which both $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are zero).

Now (1) may have several critical points. Then we discuss one after another. As a technical convenience, each time we first move the critical point P_0: (a, b) to be considered to the origin (0, 0). This can be done by a translation

$$\tilde{y}_1 = y_1 - a, \qquad \tilde{y}_2 = y_2 - b$$

which moves P_0 to (0, 0). Thus we can assume P_0 to be the origin (0, 0), and for simplicity we continue to write y_1, y_2 (instead of $\tilde{y}_1$, $\tilde{y}_2$). We also assume that P_0 is **isolated,** that is, it is the only critical point of (1) within a (sufficiently small) disk with center at the origin. If (1) has only finitely many critical points, this is automatically true. (Explain!)

Linearization of Nonlinear Systems

How can we determine the kind and stability property of a critical point P_0: (0, 0) of (1)? In most cases this can be done by **linearization** of (1) near P_0, writing (1) as $\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y})$ and dropping $\mathbf{h}(\mathbf{y})$, as follows.

Since P_0 is critical, $f_1(0, 0) = 0$, $f_2(0, 0) = 0$, so that f_1 and f_2 have no constant terms and we can write

$$(2) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y}), \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2). \end{aligned}$$

$\mathbf{A}$ is constant (independent of t) since (1) is autonomous. One can prove the following (proof in Ref. [A7], pp. 375–388, listed in App. 1).

THEOREM 1 **Linearization**

If f_1 and f_2 in (1) are continuous and have continuous partial derivatives in a neighborhood of the critical point P_0: (0, 0), and if $\det \mathbf{A} \neq 0$ *in (2), then the kind and stability of the critical point of (1) are the same as those of the* **linearized system**

$$\text{(3)} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

Exceptions occur if **A** *has equal or pure imaginary eigenvalues; then (1) may have the same kind of critical point as (3) or a spiral point.*

EXAMPLE 1 **Free Undamped Pendulum. Linearization**

Figure 92a shows a pendulum consisting of a body of mass m (the bob) and a rod of length L. Determine the locations and types of the critical points. Assume that the mass of the rod and air resistance are negligible.

Solution. ***Step 1. Setting up the mathematical model.*** Let θ denote the angular displacement, measured counterclockwise from the equilibrium position. The weight of the bob is mg (g the acceleration of gravity). It causes a restoring force $mg \sin\theta$ tangent to the curve of motion (circular arc) of the bob. By Newton's second law, at each instant this force is balanced by the force of acceleration $mL\theta''$, where $L\theta''$ is the acceleration; hence the resultant of these two forces is zero, and we obtain as the mathematical model

$$mL\theta'' + mg\sin\theta = 0.$$

Dividing this by mL, we have

$$\text{(4)} \qquad \theta'' + k\sin\theta = 0 \qquad \left(k = \frac{g}{L}\right).$$

When θ is very small, we can approximate $\sin\theta$ rather accurately by θ and obtain as an *approximate* solution $A\cos\sqrt{k}t + B\sin\sqrt{k}t$, but the *exact* solution for any θ is not an elementary function.

Step 2. Critical points **(0, 0), $\pm(2\pi, 0)$, $\pm(4\pi, 0)$, $\cdots$**, ***Linearization.*** To obtain a system of ODEs, we set $\theta = y_1$, $\theta' = y_2$. Then from (4) we obtain a nonlinear system (1) of the form

$$\text{(4*)} \qquad \begin{aligned} y_1' &= f_1(y_1, y_2) = y_2 \\ y_2' &= f_2(y_1, y_2) = -k\sin y_1. \end{aligned}$$

The right sides are both zero when $y_2 = 0$ and $\sin y_1 = 0$. This gives infinitely many critical points $(n\pi, 0)$, where $n = 0, \pm 1, \pm 2, \cdots$. We consider (0, 0). Since the Maclaurin series is

$$\sin y_1 = y_1 - \tfrac{1}{6}y_1^3 + - \cdots \approx y_1,$$

the linearized system at (0, 0) is

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1. \end{aligned}$$

To apply our criteria in Sec. 4.4 we calculate $p = a_{11} + a_{22} = 0$, $q = \det \mathbf{A} = k = g/L\ (> 0)$, and $\Delta = p^2 - 4q = -4k$. From this and Table 4.1(c) in Sec. 4.4 we conclude that (0, 0) is a center, which is always stable. Since $\sin\theta = \sin y_1$ is periodic with period 2π, the critical points $(n\pi, 0)$, $n = \pm 2, \pm 4, \cdots$, are all centers.

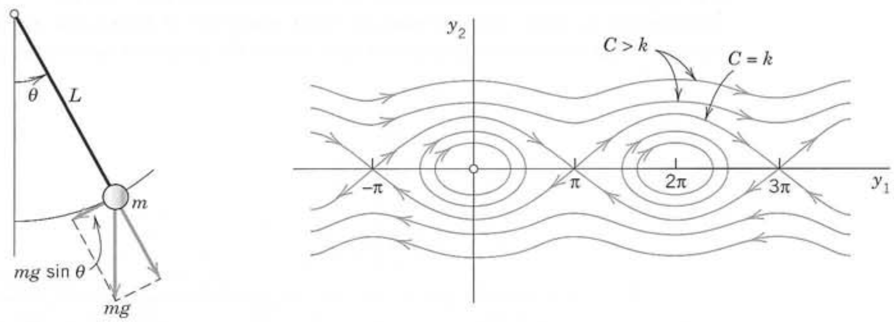
Step 3. Critical points **$\pm(\pi, 0)$, $\pm(3\pi, 0)$, $\pm(5\pi, 0)$, $\cdots$**, ***Linearization.*** We now consider the critical point $(\pi, 0)$, setting $\theta - \pi = y_1$ and $(\theta - \pi)' = \theta' = y_2$. Then in (4),

$$\sin\theta = \sin(y_1 + \pi) = -\sin y_1 = -y_1 + \tfrac{1}{6}y_1^3 - + \cdots \approx -y_1$$

and the linearized system at $(\pi, 0)$ is now

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1. \end{aligned}$$

We see that $p = 0$, $q = -k$ (< 0), and $\Delta = -4q = 4k$. Hence, by Table 4.1(b), this gives a saddle point, which is always unstable. Because of periodicity, the critical points $(n\pi, 0)$, $n = \pm 1, \pm 3, \cdots$, are all saddle points. These results agree with the impression we get from Fig. 92b. ■



(a) Pendulum

(b) Solution curves $y_2(y_1)$ of (4) in the phase plane

Fig. 92. Example 1 (*C* will be explained in Example 4.)

EXAMPLE 2 Linearization of the Damped Pendulum Equation

To gain further experience in investigating critical points, as another practically important case, let us see how Example 1 changes when we add a damping term $c\theta'$ (damping proportional to the angular velocity) to equation (4), so that it becomes

$$\theta'' + c\theta' + k \sin \theta = 0 \tag{5}$$

where $k > 0$ and $c \geqq 0$ (which includes our previous case of no damping, $c = 0$). Setting $\theta = y_1$, $\theta' = y_2$, as before, we obtain the nonlinear system (use $\theta'' = y_2'$)

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k \sin y_1 - cy_2. \end{aligned}$$

We see that the critical points have the same locations as before, namely, $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, $\cdots$. We consider $(0, 0)$. Linearizing $\sin y_1 \approx y_1$ as in Example 1, we get the linearized system at $(0, 0)$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1 - cy_2. \end{aligned} \tag{6}$$

This is identical with the system in Example 2 of Sec 4.4, except for the (positive!) factor m (and except for the physical meaning of y_1). Hence for $c = 0$ (no damping) we have a center (see Fig. 92b), for small damping we have a spiral point (see Fig. 93), and so on.

We now consider the critical point $(\pi, 0)$. We set $\theta - \pi = y_1$, $(\theta - \pi)' = \theta' = y_2$ and linearize

$$\sin \theta = \sin (y_1 + \pi) = -\sin y_1 \approx -y_1.$$

This gives the new linearized system at $(\pi, 0)$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= ky_1 - cy_2. \end{aligned} \tag{6*}$$

For our criteria in Sec 4.4 we calculate $p = a_{11} + a_{22} = -c$, $q = \det \mathbf{A} = -k$, and $\Delta = p^2 - 4q = c^2 + 4k$. This gives the following results for the critical point at $(\pi, 0)$.

No damping. $c = 0, p = 0, q < 0, \Delta > 0$, a saddle point. See Fig. 92b.
Damping. $c > 0, p < 0, q < 0, \Delta > 0$, a saddle point. See Fig. 93.

Since $\sin y_1$ is periodic with period 2π, the critical points $(\pm 2\pi, 0), (\pm 4\pi, 0), \cdots$ are of the same type as $(0, 0)$, and the critical points $(-\pi, 0), (\pm 3\pi, 0), \cdots$ are of the same type as $(\pi, 0)$, so that our task is finished.

Figure 93 shows the trajectories in the case of damping. What we see agrees with our physical intuition. Indeed, damping means loss of energy. Hence instead of the closed trajectories of periodic solutions in Fig. 92b we now have trajectories spiraling around one of the critical points $(0, 0), (\pm 2\pi, 0), \cdots$. Even the wavy trajectories corresponding to whirly motions eventually spiral around one of these points. Furthermore, there are no more trajectories that connect critical points (as there were in the undamped case for the saddle points). ■

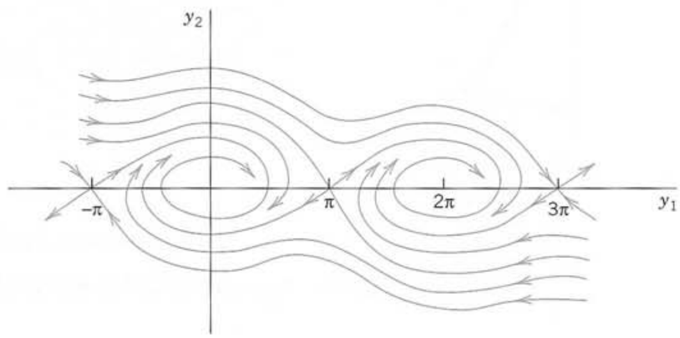


Fig. 93. Trajectories in the phase plane for the damped pendulum in Example 2

Lotka–Volterra Population Model

EXAMPLE 3 Predator–Prey Population Model[3]

This model concerns two species, say, rabbits and foxes, and the foxes prey on the rabbits.

Step 1. Setting up the model. We assume the following.

1. Rabbits have unlimited food supply. Hence if there were no foxes, their number $y_1(t)$ would grow exponentially, $y_1' = ay_1$.
2. Actually, y_1 is decreased because of the kill by foxes, say, at a rate proportional to $y_1 y_2$, where $y_2(t)$ is the number of foxes. Hence $y_1' = ay_1 - by_1y_2$, where $a > 0$ and $b > 0$.
3. If there were no rabbits, then $y_2(t)$ would exponentially decrease to zero, $y_2' = -ly_2$. However, y_2 is increased by a rate proportional to the number of encounters between predator and prey; together we have $y_2' = -ly_2 + ky_1y_2$, where $k > 0$ and $l > 0$.

This gives the (nonlinear!) Lotka–Volterra system

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = ay_1 - by_1y_2 \\ y_2' &= f_2(y_1, y_2) = ky_1y_2 - ly_2 \,. \end{aligned} \tag{7}$$

[3]Introduced by ALFRED J. LOTKA (1880–1949), American biophysicist, and VITO VOLTERRA (1860–1940), Italian mathematician, the initiator of functional analysis (see [GR7] in App. 1).

Step 2. Critical point* (0, 0), *Linearization. We see from (7) that the critical points are the solutions of

$$f_1(y_1, y_2) = y_1(a - by_2) = 0, \qquad f_2(y_1, y_2) = y_2(ky_1 - l) = 0. \tag{7*}$$

The solutions are $(y_1, y_2) = (0, 0)$ and $(\frac{l}{k}, \frac{a}{b})$. We consider (0, 0). Dropping $-by_1y_2$ and ky_1y_2 from (7) gives the linearized system

$$\mathbf{y}' = \begin{bmatrix} a & 0 \\ 0 & -l \end{bmatrix} \mathbf{y}.$$

Its eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -l < 0$. They have opposite signs, so that we get a saddle point.

Step 3. Critical point* $(l/k, a/b)$, *Linearization. We set $y_1 = \tilde{y}_1 + l/k$, $y_2 = \tilde{y}_2 + a/b$. Then the critical point $(l/k, a/b)$ corresponds to $(\tilde{y}_1, \tilde{y}_2) = (0, 0)$. Since $\tilde{y}_1' = y_1'$, $\tilde{y}_2' = y_2'$, we obtain from (7) [factorized as in (8)]

$$\tilde{y}_1' = \left(\tilde{y}_1 + \frac{l}{k}\right)\left[a - b\left(\tilde{y}_2 + \frac{a}{b}\right)\right] = \left(\tilde{y}_1 + \frac{l}{k}\right)(-b\tilde{y}_2)$$

$$\tilde{y}_2' = \left(\tilde{y}_2 + \frac{a}{b}\right)\left[k\left(\tilde{y}_1 + \frac{l}{k}\right) - l\right] = \left(\tilde{y}_2 + \frac{a}{b}\right)k\tilde{y}_1.$$

Dropping the two nonlinear terms $-b\tilde{y}_1\tilde{y}_2$ and $k\tilde{y}_1\tilde{y}_2$, we have the linearized system

$$\text{(7**)} \qquad \begin{aligned} &\text{(a)} \quad \tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2 \\ &\text{(b)} \quad \tilde{y}_2' = \frac{ak}{b}\tilde{y}_1. \end{aligned}$$

The left side of (a) times the right side of (b) must equal the right side of (a) times the left side of (b),

$$\frac{ak}{b}\tilde{y}_1\tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2\tilde{y}_2'. \qquad \text{By integration,} \qquad \frac{ak}{b}\tilde{y}_1^{\,2} + \frac{lb}{k}\tilde{y}_2^{\,2} = const.$$

This is a family ellipses, so that the critical point $(l/k, a/b)$ of the linearized system (7**) is a center (Fig. 94). It can be shown by a complicated analysis that the nonlinear system (7) also has a center (rather than a spiral point) at $(l/k, a/b)$ surrounded by closed trajectories (not ellipses).

We see that the predators and prey have a cyclic variation about the critical point. Let us move counterclockwise around the ellipse, beginning at the right vertex, where the rabbits have a maximum number. Foxes are sharply increasing in number until they reach a maximum at the upper vertex, and the number of rabbits is then sharply decreasing until it reaches a minimum at the left vertex, and so on. Cyclic variations of this kind have been observed in nature, for example, for lynx and snowshoe hare near the Hudson Bay, with a cycle of about 10 years.

For models of more complicated situations and a systematic discussion, see C. W. Clark, *Mathematical Bioeconomics* (Wiley, 1976). ■

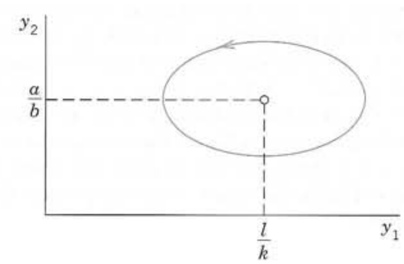


Fig. 94. Ecological equilibrium point and trajectory of the linearized Lotka–Volterra system (7**)

Transformation to a First-Order Equation in the Phase Plane

Another phase plane method is based on the idea of transforming a second-order **autonomous ODE** (an ODE in which t does not occur explicitly)

$$F(y, y', y'') = 0$$

to first order by taking $y = y_1$ as the independent variable, setting $y' = y_2$ and transforming y'' by the chain rule,

$$y'' = y_2' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1}\, y_2.$$

Then the ODE becomes of first order,

$$\textbf{(8)} \qquad F\left(y_1, y_2, \frac{dy_2}{dy_1}\, y_2\right) = 0$$

and can sometimes be solved or treated by direction fields. We illustrate this for the equation in Example 1 and shall gain much more insight into the behavior of solutions.

EXAMPLE 4 An ODE (8) for the Free Undamped Pendulum

If in (4) $\theta'' + k \sin \theta = 0$ we set $\theta = y_1$, $\theta' = y_2$ (the angular velocity) and use

$$\theta'' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1} y_2, \qquad \text{we get} \qquad \frac{dy_2}{dy_1}\, y_2 = -k \sin y_1.$$

Separation of variables gives $y_2\, dy_2 = -k \sin y_1\, dy_1$. By integration,

$$\text{(9)} \qquad \tfrac{1}{2} y_2{}^2 = k \cos y_1 + C \qquad (C \text{ constant}).$$

Multiplying this by mL^2, we get

$$\tfrac{1}{2} m(Ly_2)^2 - mL^2 k \cos y_1 = mL^2 C.$$

We see that these three terms are **energies.** Indeed, y_2 is the angular velocity, so that Ly_2 is the velocity and the first term is the kinetic energy. The second term (including the minus sign) is the potential energy of the pendulum, and mL^2C is its total energy, which is constant, as expected from the law of conservation of energy, because there is no damping (no loss of energy). The type of motion depends on the total energy, hence on C, as follows.

Figure 92b on p. 153 shows trajectories for various values of C. These graphs continue periodically with period 2π to the left and to the right. We see that some of them are ellipse-like and closed, others are wavy, and there are two trajectories (passing through the saddle points $(n\pi, 0)$, $n = \pm 1, \pm 3, \cdots$) that separate those two types of trajectories. From (9) we see that the smallest possible C is $C = -k$; then $y_2 = 0$, and $\cos y_1 = 1$, so that the pendulum is at rest. The pendulum will change its direction of motion if there are points at which $y_2 = \theta' = 0$. Then $k \cos y_1 + C = 0$ by (9). If $y_1 = \pi$, then $\cos y_1 = -1$ and $C = k$. Hence if $-k < C < k$, then the pendulum reverses its direction for a $|y_1| = |\theta| < \pi$, and for these values of C with $|C| < k$ the pendulum oscillates. This corresponds to the closed trajectories in the figure. However, if $C > k$, then $y_2 = 0$ is impossible and the pendulum makes a whirly motion that appears as a wavy trajectory in the y_1y_2-plane. Finally, the value $C = k$ corresponds to the two "separating trajectories" in Fig. 92b connecting the saddle points. ■

The phase plane method of deriving a single first-order equation (8) may be of practical interest not only when (8) can be solved (as in Example 4) but also when solution is not possible and we have to utilize direction fields (Sec. 1.2). We illustrate this with a very famous example:

EXAMPLE 5 Self-Sustained Oscillations. Van der Pol Equation

There are physical systems such that for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system. In other words, large oscillations will be damped, whereas for small oscillations there is "negative damping" (feeding of energy into the system). For physical reasons we expect such a system to approach a periodic behavior, which will thus appear as a closed trajectory in the phase plane, called a **limit cycle.** A differential equation describing such vibrations is the famous **van der Pol equation**[4]

$$y'' - \mu(1 - y^2)y' + y = 0 \qquad (\mu > 0, \text{ constant}). \tag{10}$$

It first occurred in the study of electrical circuits containing vacuum tubes. For $\mu = 0$ this equation becomes $y'' + y = 0$ and we obtain harmonic oscillations. Let $\mu > 0$. The damping term has the factor $-\mu(1 - y^2)$. This is negative for small oscillations, when $y^2 < 1$, so that we have "negative damping," is zero for $y^2 = 1$ (no damping), and is positive if $y^2 > 1$ (positive damping, loss of energy). If μ is small, we expect a limit cycle that is almost a circle because then our equation differs but little from $y'' + y = 0$. If μ is large, the limit cycle will probably look different.

Setting $y = y_1$, $y' = y_2$ and using $y'' = (dy_2/dy_1)y_2$ as in (8), we have from (10)

$$\frac{dy_2}{dy_1} y_2 - \mu(1 - y_1^2)y_2 + y_1 = 0. \tag{11}$$

The isoclines in the y_1y_2-plane (the phase plane) are the curves $dy_2/dy_1 = K = const$, that is,

$$\frac{dy_2}{dy_1} = \mu(1 - y_1^2) - \frac{y_1}{y_2} = K.$$

Solving algebraically for y_2, we see that the isoclines are given by

$$y_2 = \frac{y_1}{\mu(1 - y_1^2) - K} \qquad \text{(Figs. 95, 96).}$$

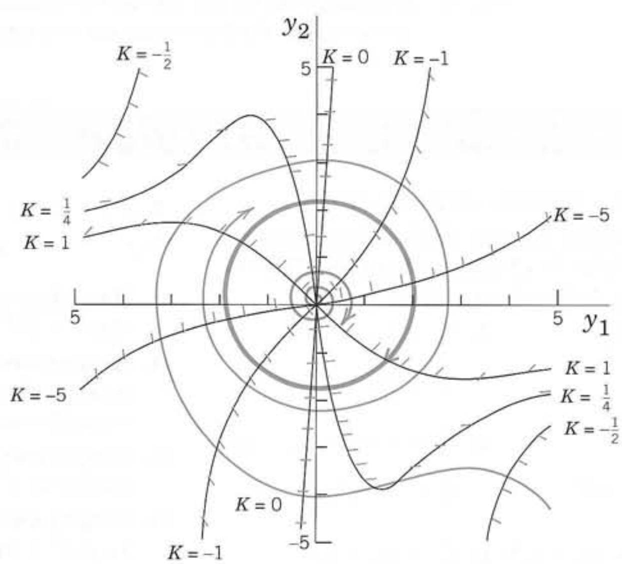


Fig. 95. Direction field for the van der Pol equation with $\mu = 0.1$ in the phase plane, showing also the limit cycle and two trajectories. See also Fig. 8 in Sec. 1.2.

[4]BALTHASAR VAN DER POL (1889–1959), Dutch physicist and engineer.

Figure 95 shows some isoclines when μ is small, $\mu = 0.1$, the limit cycle (almost a circle), and two (blue) trajectories approaching it, one from the outside and the other from the inside, of which only the initial portion, a small spiral, is shown. Due to this approach by trajectories, a limit cycle differs conceptually from a closed curve (a trajectory) surrounding a center, which is not approached by trajectories. For larger μ the limit cycle no longer resembles a circle, and the trajectories approach it more rapidly than for smaller μ. Figure 96 illustrates this for $\mu = 1$. ■

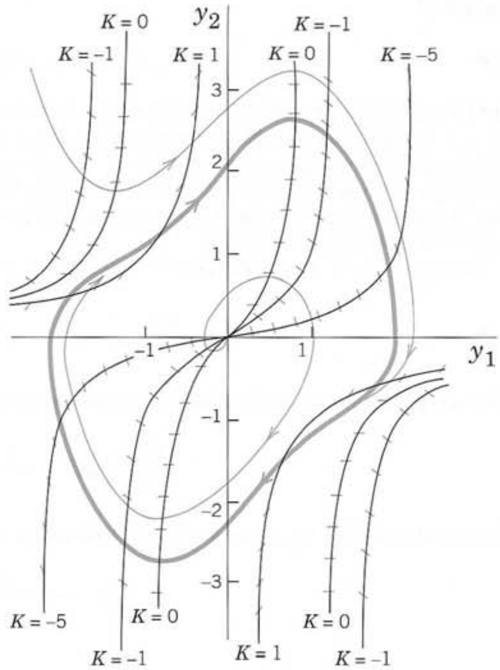


Fig. 96. Direction field for the van der Pol equation with $\mu = 1$ in the phase plane, showing also the limit cycle and two trajectories approaching it

PROBLEM SET 4.5

1–12 CRITICAL POINTS, LINEARIZATION

Determine the location and type of all critical points by linearization. In Probs. 7–12 first transform the ODE to a system. (Show the details of your work.)

1. $y_1' = y_2 + y_2^2$, $y_2' = 3y_1$

2. $y_1' = 4y_1 - y_1^2$, $y_2' = y_2$

3. $y_1' = 4y_2$, $y_2' = 2y_1 - y_1^2$

4. $y_1' = -3y_1 + y_2 - y_2^2$, $y_2' = y_1 - 3y_2$

5. $y_1' = -y_1 + y_2 - y_2^2$, $y_2' = -y_1 - y_2$

6. $y_1' = y_2 - y_2^2$, $y_2' = y_1 - y_1^2$

7. $y'' + y - 4y^2 = 0$

8. $y'' + 9y + y^2 = 0$

9. $y'' + \cos y = 0$

10. $y'' + \sin y = 0$

11. $y'' + 4y - y^3 = 0$

12. $y'' + y' + 2y - y^2 = 0$

13. **(Trajectories)** What kind of curves are the trajectories of $yy'' + 2y'^2 = 0$?

14. **(Trajectories)** Write the ODE $y'' - 4y + y^3 = 0$ as a system, solve it for y_2 as a function of y_1, and sketch or graph some of the trajectories in the phase plane.

15. **(Trajectories)** What is the radius of a real general solution of $y'' + y = 0$ in the phase plane?

16. **(Trajectories)** In Prob. 14 add a linear damping term to get $y'' + 2y' - 4y + y^3 = 0$. Using arguments from mechanics and a comparison with Prob. 14, as well as with Examples 1 and 2, guess the type of each critical point. Then determine these types by linearization. (Show all details of your work.)

17. (Pendulum) To what state (position, speed, direction of motion) do the four points of intersection of a closed trajectory with the axes in Fig. 92b correspond? The point of intersection of a wavy curve with the y_2-axis?

18. (Limit cycle) What is the essential difference between a limit cycle and a closed trajectory surrounding a center?

19. CAS EXPERIMENT. Deformation of Limit Cycle. Convert the van der Pol equation to a system. Graph the limit cycle and some approaching trajectories for $\mu = 0.2, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0$. Try to observe how the limit cycle changes its form continuously if you vary μ continuously. Describe in words how the limit cycle is deformed with growing μ.

20. TEAM PROJECT. Self-sustained oscillations.
(a) Van der Pol Equation. Determine the type of the critical point at (0, 0) when $\mu > 0$, $\mu = 0$, $\mu < 0$. Show that if $\mu \to 0$, the isoclines approach straight lines through the origin. Why is this to be expected?

(b) Rayleigh equation. Show that the so-called Rayleigh equation[5]

$$Y'' - \mu(1 - \tfrac{1}{3}Y'^2)Y' + Y = 0 \quad (\mu > 0)$$

also describes self-sustained oscillations and that by differentiating it and setting $y = Y'$ one obtains the van der Pol equation.

(c) Duffing equation. The Duffing equation is

$$y'' + \omega_0^2 y + \beta y^3 = 0$$

where usually $|\beta|$ is small, thus characterizing a small deviation of the restoring force from linearity. $\beta > 0$ and $\beta < 0$ are called the cases of a *hard spring* and a *soft spring*, respectively. Find the equation of the trajectories in the phase plane. (Note that for $\beta > 0$ all these curves are closed.)

4.6 Nonhomogeneous Linear Systems of ODEs

In this last section of Chap. 4 we discuss methods for solving nonhomogeneous linear systems of ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad (1) \qquad \text{(see Sec. 4.2)}$$

where the vector $\mathbf{g}(t)$ is not identically zero. We assume $\mathbf{g}(t)$ and the entries of the $n \times n$ matrix $\mathbf{A}(t)$ to be continuous on some interval J of the t-axis. From a general solution $\mathbf{y}^{(h)}(t)$ of the homogeneous system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ on J and a **particular solution** $\mathbf{y}^{(p)}(t)$ of (1) on J [i.e., a solution of (1) containing no arbitrary constants], we get a solution of (1),

$$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)}. \qquad (2)$$

$\mathbf{y}$ is called a **general solution** of (1) on J because it includes every solution of (1) on J. This follows from Theorem 2 in Sec. 4.2 (see Prob. 1 of this section).

Having studied homogeneous linear systems in Secs. 4.1–4.4, our present task will be to explain methods for obtaining particular solutions of (1). We discuss the method of undetermined coefficients and the method of the variation of parameters; these have counterparts for a single ODE, as we know from Secs. 2.7 and 2.10.

[5]LORD RAYLEIGH (JOHN WILLIAM STRUTT) (1842–1919), great English physicist and mathematician, professor at Cambridge and London, known by his important contributions to the theory of waves, elasticity theory, hydrodynamics, and various other branches of applied mathematics and theoretical physics. In 1904 he received the Nobel Prize in physics.

Method of Undetermined Coefficients

As for a single ODE, this method is suitable if the entries of $\mathbf{A}$ are constants and the components of $\mathbf{g}$ are constants, positive integer powers of t, exponential functions, or cosines and sines. In such a case a particular solution $\mathbf{y}^{(p)}$ is assumed in a form similar to $\mathbf{g}$; for instance, $\mathbf{y}^{(p)} = \mathbf{u} + \mathbf{v}t + \mathbf{w}t^2$ if $\mathbf{g}$ has components quadratic in t, with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to be determined by substitution into (1). This is similar to Sec. 2.7, except for the Modification Rule. It suffices to show this by an example.

EXAMPLE 1 Method of Undetermined Coefficients. Modification Rule

Find a general solution of

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}. \tag{3}$$

Solution. A general equation of the homogeneous system is (see Example 1 in Sec. 4.3)

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}. \tag{4}$$

Since $\lambda = -2$ is an eigenvalue of $\mathbf{A}$, the function e^{-2t} on the right also appears in $\mathbf{y}^{(h)}$, and we must apply the Modification Rule by setting

$$\mathbf{y}^{(p)} = \mathbf{u}te^{-2t} + \mathbf{v}e^{-2t} \qquad \text{(rather than } \mathbf{u}e^{-2t}\text{)}.$$

Note that the first of these two terms is the analog of the modification in Sec. 2.7, but it would not be sufficient here. (Try it.) By substitution,

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} - 2\mathbf{v}e^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{A}\mathbf{v}e^{-2t} + \mathbf{g}.$$

Equating the te^{-2t}-terms on both sides, we have $-2\mathbf{u} = \mathbf{A}\mathbf{u}$. Hence $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda = -2$; thus [see (5)] $\mathbf{u} = a[1 \quad 1]^\mathsf{T}$ with any $a \neq 0$. Equating the other terms gives

$$\mathbf{u} - 2\mathbf{v} = \mathbf{A}\mathbf{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \qquad \text{thus} \qquad \begin{bmatrix} a \\ a \end{bmatrix} - \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}.$$

Collecting terms and reshuffling gives

$$\begin{aligned} v_1 - v_2 &= -a - 6 \\ -v_1 + v_2 &= -a + 2. \end{aligned}$$

By addition, $0 = -2a - 4$, $a = -2$, and then $v_2 = v_1 + 4$, say, $v_1 = k$, $v_2 = k + 4$, thus, $\mathbf{v} = [k \quad k + 4]^\mathsf{T}$. We can simply choose $k = 0$. This gives the *answer*

$$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}. \tag{5}$$

For other k we get other $\mathbf{v}$; for instance, $k = -2$ gives $\mathbf{v} = [-2 \quad 2]^\mathsf{T}$, so that the *answer* becomes

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}, \qquad \text{etc.} \tag{5*}$$

■

Method of Variation of Parameters

This method can be applied to nonhomogeneous linear systems

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t) \tag{6}$$

with variable $\mathbf{A} = \mathbf{A}(t)$ and general $\mathbf{g}(t)$. It yields a particular solution $\mathbf{y}^{(p)}$ of (6) on some open interval J on the t-axis if a general solution of the homogeneous system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ on J is known. We explain the method in terms of the previous example.

EXAMPLE 2 Solution by the Method of Variation of Parameters

Solve (3) in Example 1.

Solution. A basis of solutions of the homogeneous system is $[e^{-2t} \quad e^{-2t}]^{\mathsf{T}}$ and $[e^{-4t} \quad -e^{-4t}]^{\mathsf{T}}$. Hence the general solution (4) of the homogenous system may be written

$$\mathbf{y}^{(h)} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{Y}(t)\mathbf{c}. \tag{7}$$

Here, $\mathbf{Y}(t) = [\mathbf{y}^{(1)} \quad \mathbf{y}^{(2)}]^{\mathsf{T}}$ is the fundamental matrix (see Sec. 4.2). As in Sec. 2.10 we replace the constant vector $\mathbf{c}$ by a variable vector $\mathbf{u}(t)$ to obtain a particular solution

$$\mathbf{y}^{(p)} = \mathbf{Y}(t)\mathbf{u}(t).$$

Substitution into (3) $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ gives

$$\mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' = \mathbf{A}\mathbf{Y}\mathbf{u} + \mathbf{g}. \tag{8}$$

Now since $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the homogeneous system, we have

$$\mathbf{y}^{(1)\prime} = \mathbf{A}\mathbf{y}^{(1)}, \qquad \mathbf{y}^{(2)\prime} = \mathbf{A}\mathbf{y}^{(2)}, \qquad \text{thus} \qquad \mathbf{Y}' = \mathbf{A}\mathbf{Y}.$$

Hence $\mathbf{Y}'\mathbf{u} = \mathbf{A}\mathbf{Y}\mathbf{u}$, so that (8) reduces to

$$\mathbf{Y}\mathbf{u}' = \mathbf{g}. \qquad \text{The solution is} \qquad \mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g};$$

here we use that the inverse $\mathbf{Y}^{-1}$ of $\mathbf{Y}$ (Sec. 4.0) exists because the determinant of $\mathbf{Y}$ is the Wronskian W, which is not zero for a basis. Equation (9) in Sec. 4.0 gives the form of $\mathbf{Y}^{-1}$,

$$\mathbf{Y}^{-1} = \frac{1}{-2e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}.$$

We multiply this by $\mathbf{g}$, obtaining

$$\mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 \\ -8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}.$$

Integration is done componentwise (just as differentiation) and gives

$$\mathbf{u}(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tilde{t}} \end{bmatrix} d\tilde{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

(where + 2 comes from the lower limit of integration). From this and $\mathbf{Y}$ in (7) we obtain

$$\mathbf{Y}\mathbf{u} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2te^{-2t} - 2e^{-2t} + 2e^{-4t} \\ -2te^{-2t} + 2e^{-2t} - 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -2t - 2 \\ -2t + 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}.$$

The last term on the right is a solution of the homogeneous system. Hence we can absorb it into $\mathbf{y}^{(h)}$. We thus obtain as a general solution of the system (3), in agreement with (5*),

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}. \tag{9}$$

■

PROBLEM SET 4.6

1. (General solution) Prove that (2) includes every solution of (1).

2–9 GENERAL SOLUTION

Find a general solution. (Show the details of your work.)

2. $y_1' = y_2 + t$
$y_2' = y_1 - 3t$

3. $y_1' = 4y_2 + 9t$
$y_2' = -4y_1 + 5$

4. $y_1' = y_1 + y_2 + 5\cos t$
$y_2' = 3y_1 - y_2 - 5\sin t$

5. $y_1' = 2y_1 + 2y_2 + 12$
$y_2' = 5y_1 - y_2 - 30$

6. $y_1' = -y_1 + y_2 + e^{-2t}$
$y_2' = -y_1 - y_2 - e^{-2t}$

7. $y_1' = -14y_1 + 10y_2 + 162$
$y_2' = -5y_1 + y_2 - 324t$

8. $y_1' = 10y_1 - 6y_2 + 10(1 - t - t^2)$
$y_2' = 6y_1 - 10y_2 + 4 - 20t - 6t^2$

9. $y_1' = -3y_1 - 4y_2 + 11t + 15$
$y_2' = 5y_1 + 6y_2 + 3e^{-t} - 15t - 20$

10. CAS EXPERIMENT. Undetermined Coefficients. Find out experimentally how general you must choose $\mathbf{y}^{(p)}$, in particular when the components of $\mathbf{g}$ have a different form (e.g., as in Prob. 9). Write a short report, covering also the situation in the case of the modification rule.

11–16 INITIAL VALUE PROBLEM

Solve (showing details):

11. $y_1' = -2y_2 + 4t$
$y_2' = 2y_1 - 2t$
$y_1(0) = 4,\ y_2(0) = \frac{1}{2}$

12. $y_1' = 4y_2 + 5e^t$
$y_2' = -y_1 - 20e^{-t}$
$y_1(0) = 1,\ y_2(0) = 0$

13. $y_1' = y_1 + 2y_2 + e^{2t} - 2t$
$y_2' = -y_2 + 1 + t$
$y_1(0) = 1,\ y_2(0) = -4$

14. $y_1' = 3y_1 - 4y_2 + 20\cos t$
$y_2' = y_1 - 2y_2$
$y_1(0) = 0,\ y_2(0) = 8$

15. $y_1' = 4y_2 + 3e^{3t}$
$y_2' = 2y_2 - 15e^{-3t}$
$y_1(0) = 2,\ y_2(0) = 2$

16. $y_1' = 4y_1 + 8y_2 + 2\cos t - 16\sin t$
$y_2' = 6y_1 + 2y_2 + \cos t - 14\sin t$
$y_1(0) = 15,\ y_2(0) = 13$

17. (Network) Find the currents in Fig. 97 when $R = 2.5\ \Omega$, $L = 1$ H, $C = 0.04$ F, $E(t) = 845 \sin t$ V, and $I_1(0) = 0$, $I_2(0) = 0$. (Show the details.)

18. (Network) Find the currents in Fig. 97 when $R = 1\ \Omega$, $L = 10$ H, $C = 1.25$ F, $E(t) = 10$ kV, and $I_1(0) = 0$, $I_2(0) = 0$. (Show the details.)

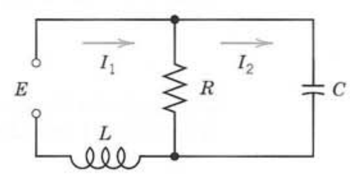


Fig. 97. Network in Probs. 17, 18

19. (Network) Find the currents in Fig. 98 when $R_1 = 2\ \Omega$, $R_2 = 8\ \Omega$, $L = 1$ H, $C = 0.5$ F, $E = 200$ V. (Show the details.)

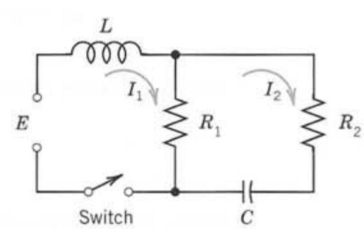


Fig. 98. Network in Prob. 19

20. WRITING PROJECT. Undetermined Coefficients. Write a short report in which you compare the application of the method of undetermined coefficients to a single ODE and to a system of two ODEs, using ODEs and systems of your choice.

CHAPTER 4 REVIEW QUESTIONS AND PROBLEMS

1. State some applications that can be modeled by systems of ODEs.

2. What is population dynamics? Give examples.

3. How can you transform an ODE into a system of ODEs?

4. What are qualitative methods for systems? Why are they important?

5. What is the phase plane? The phase plane method? The phase portrait of a system of ODEs?

6. What is a critical point of a system of ODEs? How did we classify these points?

7. What are eigenvalues? What role did they play in this chapter?

8. What does stability mean in general? In connection with critical points?

9. What does linearization of a system mean? Give an example.

10. What is a limit cycle? When may it occur in mechanics?

11–19 GENERAL SOLUTION. CRITICAL POINTS

Find a general solution. Determine the kind and stability of the critical point. (Show the details of your work.)

11. $y_1' = 4y_2$, $y_2' = 16y_1$

12. $y_1' = 9y_1$, $y_2' = y_2$

13. $y_1' = y_2$, $y_2' = 6y_1 - 5y_2$

14. $y_1' = 3y_1 - 3y_2$, $y_2' = 3y_1 + 3y_2$

15. $y_1' = 1.5y_1 - 6y_2$, $y_2' = -4.5y_1 + 3y_2$

16. $y_1' = -3y_1 - 2y_2$, $y_2' = -2y_1 - 3y_2$

17. $y_1' = 3y_1 + 2y_2$, $y_2' = 2y_1 + 3y_2$

18. $y_1' = 3y_1 + 5y_2$, $y_2' = -5y_1 - 3y_2$

19. $y_1' = -y_1 + 2y_2$, $y_2' = -2y_1 - y_2$

20–25 NONHOMOGENEOUS SYSTEMS

Find a general solution. (Show the details.)

20. $y_1' = 3y_2 + 6t$, $y_2' = 12y_1 + 1$

21. $y_1' = y_1 + 2y_2 + e^{2t}$, $y_2' = -y_2 + 1.5e^{-2t}$

22. $y_1' = y_1 + y_2 + \sin t$, $y_2' = 4y_1 + y_2$

23. $y_1' = 4y_1 + 3y_2 + 2$, $y_2' = -6y_1 - 5y_2 + 4e^{-t}$

24. $y_1' = y_1 - 2y_2 - \sin t$, $y_2' = 3y_1 - 4y_2 - \cos t$

25. $y_1' = y_1 + 2y_2 + t^2$, $y_2' = 2y_1 + y_2 - t^2$

26. **(Mixing problem)** Tank T_1 in Fig. 99 contains initially 200 gal of water in which 160 lb of salt are dissolved. Tank T_2 contains initially 100 gal of pure water. Liquid is pumped through the system as indicated, and the mixtures are kept uniform by stirring. Find the amounts of salt $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

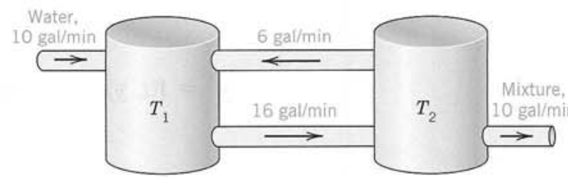


Fig. 99. Tanks in Problem 26

27. **(Critical point)** What kind of critical point does $\mathbf{y}' = \mathbf{Ay}$ have if $\mathbf{A}$ has the eigenvalues -6 and 1?

28. **(Network)** Find the currents in Fig. 100, where $R_1 = 0.5\ \Omega$, $R_2 = 0.7\ \Omega$, $L_1 = 0.4$ H, $L_2 = 0.5$ H, $E = 1$ kV $= 1000$ V, and $I_1(0) = 0$, $I_2(0) = 0$.

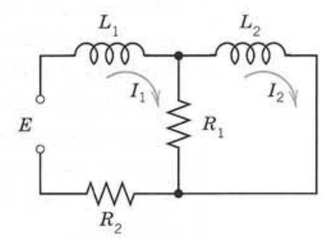


Fig. 100. Network in Problem 28

29. **(Network)** Find the currents in Fig. 101 when $R = 10\ \Omega$, $L = 1.25$ H, $C = 0.002$ F, and $I_1(0) = I_2(0) = 3$ A.

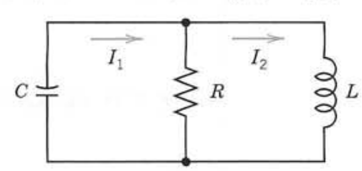


Fig. 101. Network in Problem 29

30–33 **LINEARIZATION**

Determine the location and kind of all critical points of the given nonlinear system by linearization.

30. $y_1' = y_2$
$y_2' = 4y_1 - y_1^3$

31. $y_1' = -9y_2$
$y_2' = \sin y_1$

32. $y_1' = \cos y_2$
$y_2' = 3y_1$

33. $y_1' = y_2 - 2y_2^2$
$y_2' = y_1 - 2y_1^2$

SUMMARY OF CHAPTER 4
Systems of ODEs. Phase Plane. Qualitative Methods

Whereas single electric circuits or single mass–spring systems are modeled by single ODEs (Chap. 2), networks of several circuits, systems of several masses and springs, and other engineering problems lead to **systems of ODEs,** involving several unknown functions $y_1(t), \cdots, y_n(t)$. Of central interest are **first-order systems** (Sec. 4.2):

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \qquad \text{in components,} \qquad \begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \cdots, y_n), \end{aligned}$$

to which higher order ODEs and systems of ODEs can be reduced (Sec. 4.1). In this summary we let $n = 2$, so that

$$\text{(1)} \qquad \mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \qquad \text{in components,} \qquad \begin{aligned} y_1' &= f_1(t, y_1, y_2) \\ y_2' &= f_2(t, y_1, y_2) \end{aligned}$$

Then we can represent solution curves as **trajectories** in the **phase plane** (the y_1y_2-plane), investigate their totality [the **"phase portrait"** of (1)], and study the kind and **stability** of the **critical points** (points at which both f_1 and f_2 are zero), and classify them as **nodes, saddle points, centers,** or **spiral points** (Secs. 4.3, 4.4). These phase plane methods are **qualitative;** with their use we can discover various general properties of solutions without actually solving the system. They are primarily used for **autonomous systems,** that is, systems in which t does not occur explicitly.

A **linear system** is of the form

$$\text{(2)} \quad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

If $\mathbf{g} = \mathbf{0}$, the system is called **homogeneous** and is of the form

$$\text{(3)} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y}.$$

If $a_{11}, \cdots, a_{22}$ are constants, it has solutions $\mathbf{y} = \mathbf{x}e^{\lambda t}$, where λ is a solution of the quadratic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

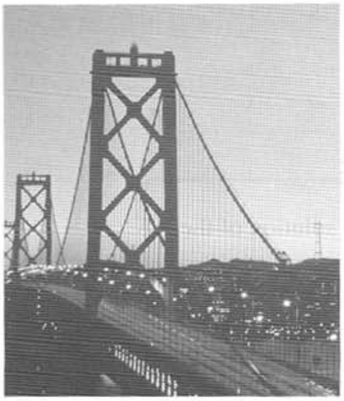
and $\mathbf{x} \neq \mathbf{0}$ has components x_1, x_2 determined up to a multiplicative constant by

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0.$$

(These λ's are called the **eigenvalues** and these vectors **x** **eigenvectors** of the matrix **A.** Further explanation is given in Sec. 4.0.)

A system (2) with $\mathbf{g} \neq \mathbf{0}$ is called **nonhomogeneous.** Its general solution is of the form $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$, where $\mathbf{y}_h$ is a general solution of (3) and $\mathbf{y}_p$ a particular solution of (2). Methods of determining the latter are discussed in Sec. 4.6.

The discussion of critical points of linear systems based on eigenvalues is summarized in Tables 4.1 and 4.2 in Sec. 4.4. It also applies to nonlinear systems if the latter are first linearized. The key theorem for this is Theorem 1 in Sec. 4.5, which also includes three famous applications, namely the pendulum and van der Pol equations and the Lotka–Volterra predator–prey population model.



CHAPTER 5

Series Solutions of ODEs. Special Functions

In Chaps. 2 and 3 we have seen that linear ODEs with *constant* coefficients can be solved by functions known from calculus. However, if a linear ODE has *variable* coefficients (functions of x), it must usually be solved by other methods, as we shall see in this chapter.

Legendre polynomials, Bessel functions, and eigenfunction expansions are the three main topics in this chapter. These are of greatest importance to the applied mathematician.

Legendre's ODE and **Legendre polynomials** (Sec. 5.3) are likely to occur in problems showing *spherical symmetry.* They are obtained by the **power series method** (Secs. 5.1, 5.2), which gives solutions of ODEs in power series.

Bessel's ODE and **Bessel functions** (Secs. 5.5, 5.6) are likely to occur in problems showing *cylindrical symmetry.* They are obtained by the **Frobenius method** (Sec. 5.4), an extension of the power series method which gives solutions of ODEs in power series, possibly multiplied by a logarithmic term or by a fractional power.

Eigenfunction expansions (Sec. 5.8) are infinite series obtained by the **Sturm–Liouville theory** (Sec. 5.7). The terms of these series may be Legendre polynomials or other functions, and their coefficients are obtained by the **orthogonality** of those functions. These expansions include **Fourier series** in terms of cosine and sine, which are so important that we shall devote a whole chapter (Chap. 11) to them.

Special functions (also called **higher functions**) is a name for more advanced functions not considered in calculus. If a function occurs in many applications, it gets a name, and its properties and values are investigated in all details, resulting in hundreds of formulas which together with the underlying theory often fill whole books. This is what has happened to the gamma, Legendre, Bessel, and several other functions (take a look into Refs. [GR1], [GR10], [A11] in App. 1).

Your CAS knows most of the special functions and corresponding formulas that you will ever need in your later work in industry, and this chapter will give you a feel for the basics of their theory and their application in modeling.

COMMENT. You can study this chapter directly after Chap. 2 because it needs no material from Chaps. 3 or 4.

Prerequisite: Chap. 2.
Sections that may be omitted in a shorter course: 5.2, 5.6–5.8.
References and Answers to Problems: App. 1 Part A, and App. 2.

5.1 Power Series Method

The **power series method** is the standard method for solving linear ODEs with ***variable*** coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we shall see. In this section we begin by explaining the idea of the power series method.

Power Series

From calculus we recall that a **power series** (in powers of $x - x_0$) is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots . \tag{1}$$

Here, x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **center** of the series. In particular, if $x_0 = 0$, we obtain a **power series in powers of x**

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \tag{2}$$

We shall assume that all variables and constants are real.

Familiar examples of power series are the Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$$

We note that the term "power series" usually refers to a series of the form (1) [or (2)] but ***does not include*** series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the Legendre and Bessel equations for integer values of the parameter.

Idea of the Power Series Method

The idea of the power series method for solving ODEs is simple and natural. We describe the practical procedure and illustrate it for two ODEs whose solution we know, so that we can see what is going on. The mathematical justification of the method follows in the next section.

For a given ODE

$$y'' + p(x)y' + q(x)y = 0$$

we first represent $p(x)$ and $q(x)$ by power series in powers of x (or of $x - x_0$ if solutions in powers of $x - x_0$ are wanted). Often $p(x)$ and $q(x)$ are polynomials, and then nothing needs to be done in this first step. Next we assume a solution in the form of a power series with unknown coefficients,

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3}$$

and insert this series and the series obtained by termwise differentiation,

$$\text{(4)} \quad \begin{aligned} &\text{(a)} \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ &\text{(b)} \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \end{aligned}$$

into the ODE. Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero, starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of (3) successively.

Let us show this for two simple ODEs that can also be solved by elementary methods, so that we would not need power series.

EXAMPLE 1 Solve the following ODE by power series. To grasp the idea, do this by hand; do not use your CAS (for which you could program the whole process).

$$y' = 2xy.$$

Solution. We insert (3) and (4a) into the given ODE, obtaining

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x(a_0 + a_1 x + a_2 x^2 + \cdots).$$

We must perform the multiplication by $2x$ on the right and can write the resulting equation conveniently as

$$\begin{aligned} &a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots \\ = \quad &\qquad 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + \cdots. \end{aligned}$$

For this equation to hold, the two coefficients of every power of x on both sides must be equal, that is,

$$a_1 = 0, \quad 2a_2 = 2a_0, \quad 3a_3 = 2a_1, \quad 4a_4 = 2a_2, \quad 5a_5 = 2a_3, \quad 6a_6 = 2a_4, \cdots.$$

Hence $a_3 = 0$, $a_5 = 0$, $\cdots$ and for the coefficients with even subscripts,

$$a_2 = a_0, \qquad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \qquad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \cdots;$$

a_0 remains arbitrary. With these coefficients the series (3) gives the following solution, which you should confirm by the method of separating variables.

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2}.$$

More rapidly, (3) and (4) give for the ODE $y' = 2xy$

$$1 \cdot a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} 2a_m x^{m+1}.$$

Now, to get the same general power on both sides, we make a **"shift of index"** on the left by setting $m = s + 2$, thus $m - 1 = s + 1$. Then a_m becomes a_{s+2} and x^{m-1} becomes x^{s+1}. Also the summation, which started with $m = 2$, now starts with $s = 0$ because $s = m - 2$. On the right we simply make a change of notation $m = s$, hence $a_m = a_s$ and $x^{m+1} = x^{s+1}$; also the summation now starts with $s = 0$. This altogether gives

$$a_1 + \sum_{s=0}^{\infty} (s + 2)a_{s+2}x^{s+1} = \sum_{s=0}^{\infty} 2a_s x^{s+1}.$$

Every occurring power of x must have the same coefficient on both sides; hence

$$a_1 = 0 \qquad \text{and} \qquad (s + 2)a_{s+2} = 2a_s \qquad \text{or} \qquad a_{s+2} = \frac{2}{s+2} a_s.$$

For $s = 0, 1, 2, \cdots$ we thus have $a_2 = (2/2)a_0$, $a_3 = (2/3)a_1 = 0$, $a_4 = (2/4)a_2$, $\cdots$ as before. ■

EXAMPLE 2 Solve

$$y'' + y = 0.$$

Solution. By inserting (3) and (4b) into the ODE we have

$$\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0.$$

To obtain the same general power on both series, we set $m = s + 2$ in the first series and $m = s$ in the second, and then we take the latter to the right side. This gives

$$\sum_{s=0}^{\infty} (s + 2)(s + 1)a_{s+2}x^s = -\sum_{s=0}^{\infty} a_s x^s.$$

Each power x^s must have the same coefficient on both sides. Hence $(s + 2)(s + 1)a_{s+2} = -a_s$. This gives the **recursion formula**

$$a_{s+2} = -\frac{a_s}{(s + 2)(s + 1)} \qquad (s = 0, 1, \cdots).$$

We thus obtain successively

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!}, \qquad a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!}$$

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}.$$

and so on. a_0 and a_1 remain arbitrary. With these coefficients the series (3) becomes

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots.$$

Reordering terms (which is permissible for a power series), we can write this in the form

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

and we recognize the familiar general solution

$$y = a_0 \cos x + a_1 \sin x.$$

Do we need the power series method for these or similar ODEs? Of course not; we used them just for explaining the idea of the method. What happens if we apply the method to an ODE not of the kind considered so far, even to an innocent-looking one such as $y'' + xy = 0$ ("Airy's equation")? We most likely end up with new special functions given by power series. And if such an ODE and its solutions are of practical (or theoretical) interest, we name and investigate them in terms of formulas and graphs and by numeric methods.

We shall discuss Legendre's, Bessel's, and the hypergeometric equations and their solutions, to mention just the most prominent of these ODEs. To do this with a good understanding, also in the light of your CAS, we first explain the power series method (and later an extension, the Frobenius method) in more detail.

PROBLEM SET 5.1

1–10 POWER SERIES METHOD: TECHNIQUE, FEATURES

Apply the power series method. Do this by hand, not by a CAS, so that you get a feel for the method, e.g., why a series may terminate, or has even powers only, or has no constant or linear terms, etc. Show the details of your work.

1. $y' - y = 0$
2. $y' + xy = 0$
3. $y'' + 4y = 0$
4. $y'' - y = 0$
5. $(2 + x)y' = y$
6. $y' + 3(1 + x^2)y = 0$
7. $y' = y + x$
8. $(x^5 + 4x^3)y' = (5x^4 + 12x^2)y$
9. $y'' - y' = 0$
10. $y'' - xy' + y = 0$

11–16 CAS PROBLEMS. INITIAL VALUE PROBLEMS

Solve the initial value problems by a power series. Graph the partial sum s of the powers up to and including x^5. Find the value of s (5 digits) at x_1.

11. $y' + 4y = 1, \quad y(0) = 1.25, \quad x_1 = 0.2$
12. $y' = 1 + y^2, \quad y(0) = 0, \quad x_1 = \frac{1}{4}\pi$
13. $y' = y - y^2, \quad y(0) = \frac{1}{2}, \quad x_1 = 1$
14. $(x - 2)y' = xy, \quad y(0) = 4, \quad x_1 = 2$
15. $y'' + 3xy' + 2y = 0, \quad y(0) = 1,$ $y'(0) = 1, \quad x_1 = 0.5$
16. $(1 - x^2)y'' - 2xy' + 30y = 0, \quad y(0) = 0,$ $y'(0) = 1.875, \quad x_1 = 0.5$
17. **WRITING PROJECT. Power Series.** Write a review (2–3 pages) on power series as they are discussed in calculus, using your own formulation and examples—do not just copy passages from calculus texts.
18. **LITERATURE PROJECT. Maclaurin Series.** Collect Maclaurin series of the functions known from calculus and arrange them systematically in a list that you can use for your work.

5.2 Theory of the Power Series Method

In the last section we saw that the power series method gives solutions of ODEs in the form of power series. In this section we justify the method mathematically as follows. We first review relevant facts on power series from calculus. Then we list the operations on power series needed in the method (differentiation, addition, multiplication, etc.). Near the end we state the basic existence theorem for power series solutions of ODEs.

Basic Concepts

Recall from calculus that a power series is an infinite series of the form

$$\textbf{(1)} \qquad \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

As before, we assume the variable x, the **center** x_0, and the **coefficients** $a_0, a_1, \cdots$ to be real. The ***n*th partial sum** of (1) is

$$\textbf{(2)} \qquad s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

where $n = 0, 1, \cdots$. Clearly, if we omit the terms of s_n from (1), the remaining expression is

$$\textbf{(3)} \qquad R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots .$$

This expression is called the **remainder** *of* (1) *after the term* $a_n(x - x_0)^n$.

For example, in the case of the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

we have

$$\begin{aligned} s_0 &= 1, & R_0 &= x + x^2 + x^3 + \cdots, \\ s_1 &= 1 + x, & R_1 &= x^2 + x^3 + x^4 + \cdots, \\ s_2 &= 1 + x + x^2, & R_2 &= x^3 + x^4 + x^5 + \cdots, \qquad \text{etc.} \end{aligned}$$

In this way we have now associated with (1) the sequence of the partial sums $s_0(x), s_1(x), s_2(x), \cdots$. If for some $x = x_1$ this sequence converges, say,

$$\lim_{n \to \infty} s_n(x_1) = s(x_1),$$

then the series (1) is called **convergent** *at* $x = x_1$, the number $s(x_1)$ is called the **value** or *sum* of (1) at x_1, and we write

$$s(x_1) = \sum_{m=0}^{\infty} a_m(x_1 - x_0)^m.$$

Then we have for every n,

$$(4) \qquad s(x_1) = s_n(x_1) + R_n(x_1).$$

If that sequence diverges at $x = x_1$, the series (1) is called **divergent** at $x = x_1$.

In the case of convergence, for any positive ϵ there is an N (depending on ϵ) such that, by (4),

$$(5) \qquad |R_n(x_1)| = |s(x_1) - s_n(x_1)| < \epsilon \qquad \text{for all } n > N.$$

Geometrically, this means that all $s_n(x_1)$ with $n > N$ lie between $s(x_1) - \epsilon$ and $s(x_1) + \epsilon$ (Fig. 102). Practically, this means that in the case of convergence we can approximate the sum $s(x_1)$ of (1) at x_1 by $s_n(x_1)$ as accurately as we please, by taking n large enough.

Convergence Interval. Radius of Convergence

With respect to the convergence of the power series (1) there are three cases, the useless Case 1, the usual Case 2, and the best Case 3, as follows.

Case 1. The series (1) always converges at $x = x_0$, because for $x = x_0$ all its terms are zero, perhaps except for the first one, a_0. In exceptional cases $x = x_0$ may be the only x for which (1) converges. Such a series is of no practical interest.

Case 2. If there are further values of x for which the series converges, these values form an interval, called the **convergence interval.** If this interval is finite, it has the midpoint x_0, so that it is of the form

(6) $$|x - x_0| < R \qquad \text{(Fig. 103)}$$

and the series (1) converges for all x such that $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$. (No general statement about convergence or divergence can be made for $x - x_0 = R$ or $-R$.) The number R is called the **radius of convergence** of (1). (R is called "radius" because for a *complex* power series it is the radius of a disk of convergence.) R can be obtained from either of the formulas

(7) $$\text{(a)}\quad R = 1\Big/\lim_{m\to\infty}\sqrt[m]{|a_m|} \qquad \text{(b)}\quad R = 1\Big/\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$$

provided these limits exist and are not zero. [If these limits are infinite, then (1) converges only at the center x_0.]

Case 3. The convergence interval may sometimes be infinite, that is, (1) converges for all x. For instance, if the limit in (7a) or (7b) is zero, this case occurs. One then writes $R = \infty$, for convenience. (Proofs of all these facts can be found in Sec. 15.2.)

For each x for which (1) converges, it has a certain value $s(x)$. We say that (1) **represents** the function $s(x)$ in the convergence interval and write

$$s(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m \qquad (|x - x_0| < R).$$

Let us illustrate these three possible cases with typical examples.

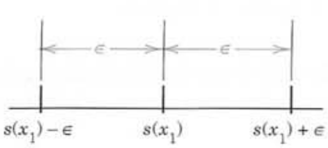


Fig. 102. Inequality (5)

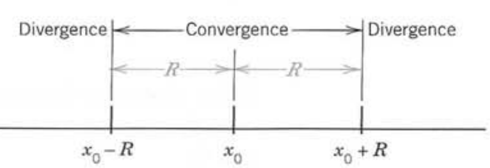


Fig. 103. Convergence interval (6) of a power series with center x_0

EXAMPLE 1 The Useless Case 1 of Convergence Only at the Center

In the case of the series

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

we have $a_m = m!$, and in (7b),

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = m + 1 \to \infty \qquad \text{as } m \to \infty.$$

Thus this series converges only at the center $x = 0$. Such a series is useless. ■

EXAMPLE 2 The Usual Case 2 of Convergence in a Finite Interval. Geometric Series

For the **geometric series** we have

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1).$$

In fact, $a_m = 1$ for all m, and from (7) we obtain $R = 1$, that is, the geometric series converges and represents $1/(1 - x)$ when $|x| < 1$. ■

EXAMPLE 3 The Best Case 3 of Convergence for All *x*

In the case of the series

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

we have $a_m = 1/m!$. Hence in (7b),

$$\frac{a_{m+1}}{a_m} = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0 \qquad \text{as } m \to \infty,$$

so that the series converges for all x. ■

EXAMPLE 4 Hint for Some of the Problems

Find the radius of convergence of the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + - \cdots.$$

Solution. This is a series in powers of $t = x^3$ with coefficients $a_m = (-1)^m/8^m$, so that in (7b),

$$\left| \frac{a_{m+1}}{a_m} \right| = \frac{8^m}{8^{m+1}} = \frac{1}{8}.$$

Thus $R = 8$. Hence the series converges for $|t| = |x^3| < 8$, that is, $|x| < 2$. ■

Operations on Power Series

In the power series method we differentiate, add, and multiply power series. These three operations are permissible, in the sense explained in what follows. We also list a condition about the vanishing of all coefficients of a power series, which is a basic tool of the power series method. (Proofs can be found in Sec. 15.3.)

Termwise Differentiation

A power series may be differentiated term by term. More precisely: if

$$y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m$$

converges for $|x - x_0| < R$, where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x, that is,

$$y'(x) = \sum_{m=1}^{\infty} m a_m(x - x_0)^{m-1} \qquad (|x - x_0| < R).$$

Similarly,

$$y''(x) = \sum_{m=2}^{\infty} m(m-1)a_m(x - x_0)^{m-2} \qquad (|x - x_0| < R), \text{ etc.}$$

Termwise Addition

Two power series may be added term by term. More precisely: if the series

$$\sum_{m=0}^{\infty} a_m(x - x_0)^m \qquad \text{and} \qquad \sum_{m=0}^{\infty} b_m(x - x_0)^m \tag{8}$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then the series

$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges and represents $f(x) + g(x)$ for each x that lies in the interior of the convergence interval of each of the two given series.

Termwise Multiplication

Two power series may be multiplied term by term. More precisely: Suppose that the series (8) have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$, that is,

$$\sum_{m=0}^{\infty} (a_0b_m + a_1b_{m-1} + \cdots + a_mb_0)(x - x_0)^m$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 + \cdots$$

converges and represents $f(x)g(x)$ for each x in the interior of the convergence interval of each of the two given series.

Vanishing of All Coefficients

If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

Existence of Power Series Solutions of ODEs. Real Analytic Functions

The properties of power series just discussed form the foundation of the power series method. The remaining question is whether an ODE has power series solutions at all. An answer is simple: If the coefficients p and q and the function r on the right side of

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

have power series representations, then (9) has power series solutions. The same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x) \tag{10}$$

have power series representations and $\tilde{h}(x_0) \neq 0$ (x_0 the center of the series). Almost all ODEs in practice have polynomials as coefficients (thus terminating power series), so that (when $r(x) \equiv 0$ or is a power series, too) those conditions are satisfied, except perhaps the condition $\tilde{h}(x_0) \neq 0$. If $\tilde{h}(x_0) \neq 0$, division of (10) by $\tilde{h}(x)$ gives (9) with $p = \tilde{p}/\tilde{h}$, $q = \tilde{q}/\tilde{h}$, $r = \tilde{r}/\tilde{h}$. This motivates our notation in (10).

To formulate all this in a precise and simple way, we use the following concept (which is of general interest).

DEFINITION

Real Analytic Function

A real function $f(x)$ is called **analytic** *at a point* $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Using this concept, we can state the following basic theorem.

THEOREM 1

Existence of Power Series Solutions

If p, q, and r in (9) are analytic at $x = x_0$, then every solution of (9) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. Hence the same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in (10) are analytic at $x = x_0$ and $\tilde{h}(x_0) \neq 0$.

The proof of this theorem requires advanced methods of complex analysis and can be found in Ref. [A11] listed in App. 1.

We mention that the radius of convergence R in Theorem 1 is at least equal to the distance from the point $x = x_0$ to the point (or points) closest to x_0 at which one of the functions p, q, r, as functions of a *complex variable,* is not analytic. (Note that that point may not lie on the x-axis but somewhere in the complex plane.)

PROBLEM SET 5.2

1–12 RADIUS OF CONVERGENCE

Determine the radius of convergence. (Show the details.)

1. $\sum_{m=0}^{\infty} \frac{x^m}{c^m} \quad (c \neq 0)$

2. $\sum_{m=0}^{\infty} \frac{(-1)^m}{3^m(m+1)^2} (x+1)^{2m}$

3. $\sum_{m=1}^{\infty} \frac{(m+1)m}{2^m} (x-3)^{2m}$

4. $\sum_{m=0}^{\infty} (-1)^m x^{4m}$

5. $\sum_{m=0}^{\infty} \frac{(2m)!}{(2m+2)(2m+4)} x^m$

6. $\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} x^{2m+10}$

7. $\sum_{m=2}^{\infty} \frac{(-1)^m}{4^m} (x-1)^{2m}$

8. $\sum_{m=1}^{\infty} \frac{(4m)!}{(m!)^4} x^m$

9. $\sum_{m=4}^{\infty} \frac{(m+3)^2}{(m-3)^4} x^m$

10. $\sum_{m=1}^{\infty} \frac{(2m)!}{m^2} x^m$

11. $\sum_{m=1}^{\infty} \frac{1}{\pi^m} (x - \frac{1}{2}\pi)^m$

12. $\sum_{m=1}^{\infty} \frac{(m+1)m}{(2m+1)!} x^{2m+1}$

13–15 SHIFTING SUMMATION INDICES (CF. SEC. 5.1)

This is often convenient or necessary in the power series method. Shift the index so that the power under the summation sign is x^s. Check by writing the first few terms explicitly. Also determine the radius of convergence R.

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n} x^{n+2}$

14. $\sum_{m=3}^{\infty} \frac{(-1)^{m+1}}{4^m} x^{m-3}$

15. $\sum_{p=1}^{\infty} \frac{p^2}{(p+1)!} x^{p+4}$

16–23 POWER SERIES SOLUTIONS

Find a power series solution in powers of x. (Show the details of your work.)

16. $y'' + xy = 0$
17. $y'' - y' + x^2y = 0$
18. $y'' - y' + xy = 0$
19. $y'' + 4xy' = 0$
20. $y'' + 2xy' + y = 0$
21. $y'' + (1 + x^2)y = 0$
22. $y'' - 4xy' + (4x^2 - 2)y = 0$
23. $(2x^2 - 3x + 1)y'' + 2xy' - 2y = 0$

24. **TEAM PROJECT. Properties from Power Series.** In the next sections we shall define new functions (Legendre functions, etc.) by power series, deriving properties of the functions directly from the series. To understand this idea, do the same for functions familiar from calculus, using Maclaurin series.

(a) Show that $\cosh x + \sinh x = e^x$. Show that $\cosh x > 0$ for all x. Show that $e^x \geqq e^{-x}$ for all $x \geqq 0$.

(b) Derive the differentiation formulas for e^x, $\cos x$, $\sin x$, $1/(1-x)$ and other functions of your choice. Show that $(\cos x)'' = -\cos x$, $(\cosh x)'' = \cosh x$. Consider integration similarly.

(c) What can you conclude if a series contains only odd powers? Only even powers? No constant term? If all its coefficients are positive? Give examples.

(d) What properties of $\cos x$ and $\sin x$ are *not* obvious from the Maclaurin series? What properties of other functions?

25. **CAS EXPERIMENT. Information from Graphs of Partial Sums.** In connection with power series in numerics we use partial sums. To get a feel for the accuracy for various x, experiment with $\sin x$ and graphs of partial sums of the Maclaurin series of an increasing number of terms, describing qualitatively the "breakaway points" of these graphs from the graph of $\sin x$. Consider other examples of your own choice.

5.3 Legendre's Equation. Legendre Polynomials $P_n(x)$

In order to first gain skill, we have applied the power series method to ODEs that can also be solved by other methods. We now turn to the first "big" equation of physics, for which we do need the power series method. This is **Legendre's equation**[1]

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{1}$$

where n is a given constant. Legendre's equation arises in numerous problems, particularly in boundary value problems for spheres (take a quick look at Example 1 in Sec. 12.10). The **parameter** n in (1) is a given real number. Any solution of (1) is called a **Legendre function.** The study of these and other "higher" functions not occurring in calculus is called the **theory of special functions.** Further special functions will occur in the next sections.

Dividing (1) by the coefficient $1 - x^2$ of y'', we see that the coefficients $-2x/(1 - x^2)$ and $n(n + 1)/(1 - x^2)$ of the new equation are analytic at $x = 0$. Hence by Theorem 1, in Sec. 5.2, Legendre's equation has power series solutions of the form

$$y = \sum_{m=0}^{\infty} a_m x^m. \tag{2}$$

Substituting (2) and its derivatives into (1), and denoting the constant $n(n + 1)$ simply by k, we obtain

$$(1 - x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x^s in all four series, we set $m - 2 = s$ (thus $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

---

[1] ADRIEN-MARIE LEGENDRE (1752–1833), French mathematician, who became a professor in Paris in 1775 and made important contributions to special functions, elliptic integrals, number theory, and the calculus of variations, His book *Éléments de géométrie* (1794) became very famous and had 12 editions in less than 30 years.

Formulas on Legendre functions may be found in Refs. [GR1] and [GR10].

(Note that in the first series the summation begins with $s = 0$.) Since this equation with right side 0 must be an identity in x if (2) is to be a solution of (1), the sum of the coefficients of each power of x on the left must be zero. Now x^0 occurs in the first and fourth series and gives [remember that $k = n(n+1)$]

$$2 \cdot 1 a_2 + n(n+1)a_0 = 0. \tag{3a}$$

x^1 occurs in the first, third, and fourth series and gives

$$3 \cdot 2 a_3 + [-2 + n(n+1)]a_1 = 0. \tag{3b}$$

The higher powers $x^2, x^3, \cdots$ occur in all four series and give

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0. \tag{3c}$$

The expression in the brackets $[\cdots]$ can be written $(n-s)(n+s+1)$, as you may readily verify. Solving (3a) for a_2 and (3b) for a_3 as well as (3c) for a_{s+2}, we obtain the general formula

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\, a_s \qquad (s = 0, 1, \cdots). \tag{4}$$

This is called a **recurrence relation** or **recursion formula.** (Its derivation you may verify with your CAS.) It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively

$$a_2 = -\frac{n(n+1)}{2!}\, a_0 \qquad\qquad a_3 = -\frac{(n-1)(n+2)}{3!}\, a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3}\, a_2 \qquad\qquad a_5 = -\frac{(n-3)(n+4)}{5 \cdot 4}\, a_3$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!}\, a_0 \qquad\qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, a_1$$

and so on. By inserting these expressions for the coefficients into (2) we obtain

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{5}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots \tag{6}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots. \tag{7}$$

These series converge for $|x| < 1$ (see Prob. 4; or they may terminate, see below). Since (6) contains even powers of x only, while (7) contains odd powers of x only, the ratio y_1/y_2 is not a constant, so that y_1 and y_2 are not proportional and are thus linearly independent solutions. Hence (5) is a general solution of (1) on the interval $-1 < x < 1$.

Legendre Polynomials $P_n(x)$

In various applications, power series solutions of ODEs reduce to polynomials, that is, they terminate after finitely many terms. This is a great advantage and is quite common for special functions, leading to various important families of polynomials (see Refs. [GR1] or [GR10] in App. 1). For Legendre's equation this happens when the parameter n is a nonnegative integer because then the right side of (4) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0, \cdots$. Hence if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of a constant is done as follows. We choose the coefficient a_n of the highest power x^n as

$$(8) \qquad a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad (n \text{ a positive integer})$$

(and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from (4), solved for a_s in terms of a_{s+2}, that is,

$$(9) \qquad a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leqq n-2).$$

The choice (8) makes $P_n(1) = 1$ for every n (see Fig. 104 on p. 180); this motivates (8). From (9) with $s = n - 2$ and (8) we obtain

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}.$$

Using $(2n)! = 2n(2n-1)(2n-2)!$, $n! = n(n-1)!$, and $n! = n(n-1)(n-2)!$, we obtain

$$a_{n-2} = -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!\, n(n-1)(n-2)!}.$$

$n(n-1)2n(2n-1)$ cancels, so that we get

$$a_{n-2} = -\frac{(2n-2)!}{2^n(n-1)!\,(n-2)!}.$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!}$$

and so on, and in general, when $n - 2m \geqq 0$,

$$\textbf{(10)} \qquad a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}.$$

The resulting solution of Legendre's differential equation (1) is called the **Legendre polynomial** *of degree n* and is denoted by $P_n(x)$.

From (10) we obtain

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}$$

(11)

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!\,(n-1)!\,(n-2)!} x^{n-2} + - \cdots$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are (Fig. 104)

$$P_0(x) = 1, \qquad P_1(x) = x$$

(11′)

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1), \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

and so on. You may now program (11) on your CAS and calculate $P_n(x)$ as needed.

The so-called **orthogonality** of the Legendre polynomials will be considered in Secs. 5.7 and 5.8.

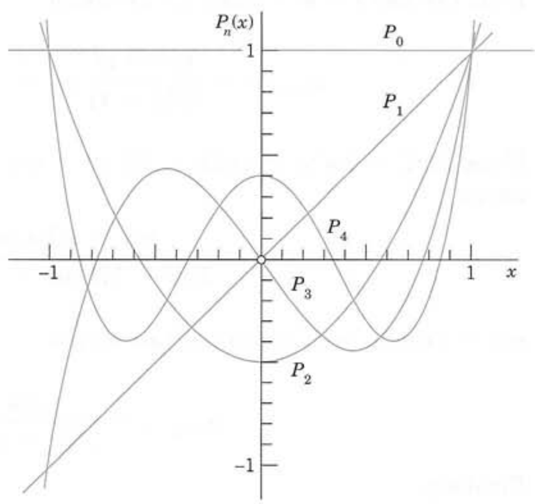


Fig. 104. Legendre polynomials

PROBLEM SET 5.3

1. Verify that the polynomials in (11′) satisfy Legendre's equation.
2. Derive (11′) from (11).
3. Obtain P_6 and P_7 from (11).
4. **(Convergence)** Show that for any n for which (6) or (7) does not reduce to a polynomial, the series has radius of convergence 1.
5. **(Legendre function $Q_0(x)$ for $n = 0$)** Show that (6) with $n = 0$ gives $y_1(x) = P_0(x) = 1$ and (7) gives

$$y_2(x) = x + \frac{2}{3!} x^3 + \frac{(-3)(-1)\cdot 2\cdot 4}{5!} x^5 + \cdots$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

Verify this by solving (1) with $n = 0$, setting $z = y'$ and separating variables.

6. (Legendre function $-Q_1(x)$ for $n = 1$) Show that (7) with $n = 1$ gives $y_2(x) = P_1(x) = x$ and (6) gives $y_1(x) = -Q_1(x)$ (the minus sign in the notation being conventional),

$$\begin{aligned} y_1(x) &= 1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots \\ &= 1 - x\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \\ &= 1 - \frac{1}{2}\, x \ln \frac{1+x}{1-x}\,. \end{aligned}$$

7. (ODE) Find a solution of $(a^2 - x^2)y'' - 2xy' + n(n+1)y = 0$, $a \neq 0$, by reduction to the Legendre equation.

8. [Rodrigues's formula (12)][2] Applying the binomial theorem to $(x^2 - 1)^n$, differentiating it n times term by term, and comparing the result with (11), show that

$$\textbf{(12)} \qquad P_n(x) = \frac{1}{2^n n!}\, \frac{d^n}{dx^n}\left[(x^2 - 1)^n\right].$$

9. (Rodrigues's formula) Obtain (11′) from (12).

10–13 CAS PROBLEMS

10. Graph $P_2(x), \cdots, P_{10}(x)$ on common axes. For what x (approximately) and $n = 2, \cdots, 10$ is $|P_n(x)| < \frac{1}{2}$?

11. From what n on will your CAS no longer produce faithful graphs of $P_n(x)$? Why?

12. Graph $Q_0(x)$, $Q_1(x)$, and some further Legendre functions.

13. Substitute $a_s x^s + a_{s+1}x^{s+1} + a_{s+2}x^{s+2}$ into Legendre's equation and obtain the coefficient recursion (4).

14. TEAM PROJECT. Generating Functions. Generating functions play a significant role in modern applied mathematics (see [GR5]). The idea is simple. If we want to study a certain sequence $(f_n(x))$ and can find a function

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x)u^n,$$

we may obtain properties of $(f_n(x))$ from those of G, which "generates" this sequence and is called a **generating function** of the sequence.

(a) Legendre polynomials. Show that

$$\textbf{(13)} \qquad G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$$

is a generating function of the Legendre polynomials. *Hint:* Start from the binomial expansion of $1/\sqrt{1 - v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

(b) Potential theory. Let A_1 and A_2 be two points in space (Fig. 105, $r_2 > 0$). Using (13), show that

$$\begin{aligned} \frac{1}{r} &= \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}} \\ &= \frac{1}{r_2} \sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m. \end{aligned}$$

This formula has applications in potential theory. (Q/r is the electrostatic potential at A_2 due to a charge Q located at A_1. And the series expresses $1/r$ in terms of the distances of A_1 and A_2 from any origin O and the angle θ between the segments OA_1 and OA_2.)

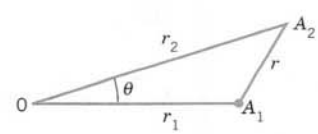


Fig. 105. Team Project 14

(c) Further applications of (13). Show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and

$$P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n-1)/[2 \cdot 4 \cdots (2n)].$$

(d) Bonnet's recursion.[3] Differentiating (13) with respect to u, using (13) in the resulting formula, and comparing coefficients of u^n, obtain the *Bonnet recursion*

$$(14) \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

where $n = 1, 2, \cdots$. This formula is useful for computations, the loss of significant digits being small (except near zeros). Try (14) out for a few computations of your own choice.

[2] OLINDE RODRIGUES (1794–1851), French mathematician and economist.

[3] OSSIAN BONNET (1819–1892), French mathematician, whose main work was in differential geometry.

15. (Associated Legendre functions) The associated Legendre functions $P_n{}^k(x)$ play a role in quantum physics. They are defined by

$$P_n{}^k(x) = (1 - x^2)^{k/2} \frac{d^k P_n}{dx^k} \tag{15}$$

and are solutions of the ODE

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{k^2}{1 - x^2} \right] y = 0. \tag{16}$$

Find $P_1{}^1(x)$, $P_2{}^1(x)$, $P_2{}^2(x)$, and $P_4{}^2(x)$ and verify that they satisfy (16).

5.4 Frobenius Method

Several second-order ODEs of considerable practical importance—the famous Bessel equation among them—have coefficients that are not analytic (definition in Sec. 5.2), but are "not too bad," so that these ODEs can still be solved by series (power series times a logarithm or times a fractional power of x, etc.). Indeed, the following theorem permits an extension of the power series method that is called the **Frobenius method.** The latter—as well as the power series method itself—has gained in significance due to the use of software in the actual calculations.

THEOREM 1

Frobenius Method

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{1}$$

has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{2}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE (1) also has a second solution (such that these two solutions are linearly independent) that may be similar to (2) (with a different r and different coefficients) or may contain a logarithmic term. (Details in Theorem 2 below.)[4]

For example, Bessel's equation (to be discussed in the next section)

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad (\nu \text{ a parameter})$$

[4]GEORG FROBENIUS (1849–1917), German mathematician, also known for his work on matrices and in group theory.

In this theorem we may replace x by $x - x_0$ with any number x_0. The condition $a_0 \neq 0$ is no restriction; it simply means that we factor out the highest possible power of x.

The singular point of (1) at $x = 0$ is sometimes called a **regular singular point,** a term confusing to the student, which we shall not use.

is of the form (1) with $b(x) = 1$ and $c(x) = x^2 - \nu^2$ analytic at $x = 0$, so that the theorem applies. This ODE could not be handled in full generality by the power series method.

Similarly, the so-called hypergeometric differential equation (see Problem Set 5.4) also requires the Frobenius method.

The point is that in (2) we have a power series times a single power of x whose exponent r is not restricted to be a nonnegative integer. (The latter restriction would make the whole expression a power series, by definition; see Sec. 5.1.)

The proof of the theorem requires advanced methods of complex analysis and can be found in Ref. [A11] listed in App. 1.

Regular and Singular Points

The following commonly used terms are practical. A **regular point** of

$$y'' + p(x)y' + q(x)y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Then the power series method can be applied. If x_0 is not regular, it is called **singular.** Similarly, a **regular point** of the ODE

$$\widetilde{h}(x)y'' + \widetilde{p}(x)y'(x) + \widetilde{q}(x)y = 0$$

is an x_0 at which $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$ are analytic and $\widetilde{h}(x_0) \neq 0$ (so what we can divide by $\widetilde{h}$ and get the previous standard form). If x_0 is not regular, it is called **singular.**

Indicial Equation, Indicating the Form of Solutions

We shall now explain the Frobenius method for solving (1). Multiplication of (1) by x^2 gives the more convenient form

$$x^2y'' + xb(x)y' + c(x)y = 0. \tag{1'}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1x + b_2x^2 + \cdots, \qquad c(x) = c_0 + c_1x + c_2x^2 + \cdots$$

or we do nothing if $b(x)$ and $c(x)$ are polynomials. Then we differentiate (2) term by term, finding

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m + r)a_m x^{m+r-1} = x^{r-1}\,[ra_0 + (r + 1)a_1x + \cdots] \\ y''(x) &= \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2} \\ &= x^{r-2}[r(r - 1)a_0 + (r + 1)ra_1x + \cdots]. \end{aligned} \tag{2*}$$

By inserting all these series into (1′) we readily obtain

$$\begin{aligned} &x^r[r(r - 1)a_0 + \cdots] + (b_0 + b_1x + \cdots)x^r(ra_0 + \cdots) \\ &\qquad + (c_0 + c_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0. \end{aligned} \tag{3}$$

We now equate the sum of the coefficients of each power $x^r, x^{r+1}, x^{r+2}, \cdots$ to zero. This yields a system of equations involving the unknown coefficients a_m. The equation corresponding to the power x^r is

$$[r(r-1) + b_0 r + c_0]a_0 = 0.$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives

$$r(r-1) + b_0 r + c_0 = 0. \tag{4}$$

This important quadratic equation is called the **indicial equation** of the ODE (1). Its role is as follows.

The Frobenius method yields a basis of solutions. One of the two solutions will always be of the form (2), where r is a root of (4). The other solution will be of a form indicated by the indicial equation. There are three cases:

Case 1. Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2. A double root.

Case 3. Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are not unexpected because of the Euler–Cauchy equation (Sec. 2.5), the simplest ODE of the form (1). Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i \operatorname{Im} r_1$ is imaginary, so it cannot be a *real* integer. The form of a basis will be given in Theorem 2 (which is proved in App. 4), without a general theory of convergence, but convergence of the occurring series can be tested in each individual case as usual. Note that in Case 2 we ***must*** have a logarithm, whereas in Case 3 we *may* or *may not.*

THEOREM 2

Frobenius Method. Basis of Solutions. Three Cases

Suppose that the ODE (1) *satisfies the assumptions in Theorem* 1. *Let* r_1 *and* r_2 *be the roots of the indicial equation* (4). *Then we have the following three cases.*

Case* 1. *Distinct Roots Not Differing by an Integer. *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{6}$$

with coefficients obtained successively from (3) *with* $r = r_1$ *and* $r = r_2$, *respectively.*

Case* 2. *Double Root $r_1 = r_2 = r$. *A basis is*

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \left[r = \tfrac{1}{2}(1 - b_0)\right] \tag{7}$$

(*of the same general form as before*) *and*

$$y_2(x) = y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0). \tag{8}$$

Case 3. Roots Differing by an Integer. *A basis is*

(9) $$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(of the same general form as before) and

(10) $$y_2(x) = ky_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

Typical Applications

Technically, the Frobenius method is similar to the power series method, once the roots of the indicial equation have been determined. However, (5)–(10) merely indicate the general form of a basis, and a second solution can often be obtained more rapidly by reduction of order (Sec. 2.1).

EXAMPLE 1 Euler–Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm

For the Euler–Cauchy equation (Sec. 2.5)

$$x^2y'' + b_0xy' + c_0y = 0 \qquad (b_0, c_0 \text{ constant})$$

substitution of $y = x^r$ gives the auxiliary equation

$$r(r-1) + b_0 r + c_0 = 0,$$

which is the indicial equation [and $y = x^r$ is a very special form of (2)!]. For different roots r_1, r_2 we get a basis $y_1 = x^{r_1}$, $y_2 = x^{r_2}$, and for a double root r we get a basis x^r, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role. ∎

EXAMPLE 2 Illustration of Case 2 (Double Root)

Solve the ODE

(11) $$x(x-1)y'' + (3x-1)y' + y = 0.$$

(This is a special hypergeometric equation, as we shall see in the problem set.)

Solution. Writing (11) in the standard form (1), we see that it satisfies the assumptions in Theorem 1. [What are $b(x)$ and $c(x)$ in (11)?] By inserting (2) and its derivatives (2*) into (11) we obtain

(12) $$\begin{aligned}&\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-1} \\ &+ 3\sum_{m=0}^{\infty} (m+r)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.\end{aligned}$$

The smallest power is x^{r-1}, occurring in the second and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]a_0 = 0, \qquad \text{thus} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.

First Solution. We insert this value $r = 0$ into (12) and equate the sum of the coefficients of the power x^s to zero, obtaining

$$s(s-1)a_s - (s+1)sa_{s+1} + 3sa_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

Second Solution. We get a second independent solution y_2 by the method of reduction of order (Sec. 2.1), substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9), Sec. 2.1, which we shall use in this example, instead of starting reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of y' in (11) ***in standard form.*** By partial fractions,

$$-\int p\,dx = -\int \frac{3x-1}{x(x-1)}\,dx = -\int \left(\frac{2}{x-1} + \frac{1}{x}\right) dx = -2\ln(x-1) - \ln x.$$

Hence (9), Sec. 2.1, becomes

$$u' = U = y_1^{-2}e^{-\int p\,dx} = \frac{(x-1)^2}{(x-1)^2x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = uy_1 = \frac{\ln x}{1-x}.$$

y_1 and y_2 are shown in Fig. 106. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$). ■

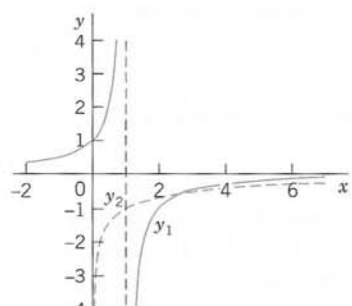


Fig. 106. Solutions in Example 2

EXAMPLE 3 Case 3, Second Solution with Logarithmic Term

Solve the ODE

$$(x^2 - x)y'' - xy' + y = 0. \tag{13}$$

Solution. Substituting (2) and (2*) into (13), we have

$$(x^2-x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} - x\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0.$$

We now take x^2, x, and x inside the summations and collect all terms with power x^{m+r} and simplify algebraically,

$$\sum_{m=0}^{\infty}(m+r-1)^2 a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} = 0.$$

In the first series we set $m = s$ and in the second $m = s+1$, thus $s = m-1$. Then

$$\sum_{s=0}^{\infty}(s+r-1)^2 a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+r+1)(s+r)a_{s+1}x^{s+r} = 0. \tag{14}$$

The lowest power is x^{r-1} (take $s = -1$ in the second series) and gives the indicial equation

$$r(r-1) = 0.$$

The roots are $r_1 = 1$ and $r_2 = 0$. They differ by an integer. This is Case 3.

First Solution. From (14) with $r = r_1 = 1$ we have

$$\sum_{s=0}^{\infty} \left[s^2 a_s - (s+2)(s+1)a_{s+1}\right]x^{s+1} = 0.$$

This gives the recurrence relation

$$a_{s+1} = \frac{s^2}{(s+2)(s+1)}\, a_s \qquad (s = 0, 1, \cdots).$$

Hence $a_1 = 0, a_2 = 0, \cdots$ successively. Taking $a_0 = 1$, we get as a first solution $y_1 = x^{r_1}a_0 = x$.

Second Solution. Applying reduction of order (Sec. 2.1), we substitute $y_2 = y_1 u = xu$, $y_2' = xu' + u$ and $y_2'' = xu'' + 2u'$ into the ODE, obtaining

$$(x^2 - x)(xu'' + 2u') - x(xu' + u) + xu = 0.$$

xu drops out. Division by x and simplification give

$$(x^2 - x)u'' + (x - 2)u' = 0.$$

From this, using partial fractions and integrating (taking the integration constant zero), we get

$$\frac{u''}{u'} = -\frac{x-2}{x^2 - x} = -\frac{2}{x} + \frac{1}{x-1}, \qquad \ln u' = \ln\left|\frac{x-1}{x^2}\right|.$$

Taking exponents and integrating (again taking the integration constant zero), we obtain

$$u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \ln x + \frac{1}{x}, \qquad y_2 = xu = x\ln x + 1.$$

y_1 and y_2 are linearly independent, and y_2 has a logarithmic term. Hence y_1 and y_2 constitute a basis of solutions for positive x. ■

The Frobenius method solves the **hypergeometric equation,** whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

PROBLEM SET 5.4

1–17 BASIS OF SOLUTIONS BY THE FROBENIUS METHOD

Find a basis of solutions. Try to identify the series as expansions of known functions. (Show the details of your work.)

1. $xy'' + 2y' - xy = 0$
2. $(x+2)^2 y'' - 2y = 0$
3. $xy'' + 5y' + xy = 0$
4. $2xy'' + (3 - 4x)y' + (2x - 3)y = 0$
5. $x^2y'' + 4xy' + (x^2 + 2)y = 0$
6. $4xy'' + 2y' + y = 0$
7. $(x+3)^2y'' - 9(x+3)y' + 25y = 0$
8. $xy'' - y = 0$
9. $xy'' + (2x+1)y' + (x+1)y = 0$
10. $x^2y'' + 2x^3y' + (x^2 - 2)y = 0$
11. $(x^2 + x)y'' + (4x + 2)y' + 2y = 0$
12. $x^2y'' + 6xy' + (4x^2 + 6)y = 0$
13. $2xy'' - (8x - 1)y' + (8x - 2)y = 0$
14. $xy'' + y' - xy = 0$
15. $(x-4)^2y'' - (x-4)y' - 35y = 0$
16. $x^2y'' + 4xy' - (x^2 - 2)y = 0$
17. $y'' + (x - 6)y = 0$

18. TEAM PROJECT. Hypergeometric Equation, Series, and Function. Gauss's **hypergeometric ODE**[5] is

$$(15)\quad x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0.$$

Here, a, b, c are constants. This ODE is of the form $p_2y'' + p_1y' + p_0y = 0$, where p_2, p_1, p_0 are polynomials of degree 2, 1, 0, respectively. These polynomials are written so that the series solution takes a most practical form, namely,

$$(16)\quad \begin{aligned} y_1(x) &= 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 \\ &\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots. \end{aligned}$$

This series is called the **hypergeometric series.** Its sum $y_1(x)$ is called the **hypergeometric function** and is denoted by $F(a, b, c; x)$. Here, $c \neq 0, -1, -2, \cdots$. By choosing specific values of a, b, c we can obtain an incredibly large number of special functions as solutions of (15) [see the small sample of elementary functions in part (c)]. This accounts for the importance of (15).

(a) Hypergeometric series and function. Show that the indicial equation of (15) has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method gives (16). Motivate the name for (16) by showing that

$$F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x) = \frac{1}{1-x}.$$

(b) Convergence. For what a or b will (16) reduce to a polynomial? Show that for any other a, b, c ($c \neq 0, -1, -2, \cdots$) the series (16) converges when $|x| < 1$.

(c) Special cases. Show that

$$(1+x)^n = F(-n, b, b; -x),$$
$$(1-x)^n = 1 - nxF(1-n, 1, 2; x),$$
$$\arctan x = xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2),$$
$$\arcsin x = xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2),$$
$$\ln(1+x) = xF(1, 1, 2; -x),$$
$$\ln\frac{1+x}{1-x} = 2xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2).$$

Find more such relations from the literature on special functions.

(d) Second solution. Show that for $r_2 = 1 - c$ the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$(17)\quad \begin{aligned} y_2(x) = x^{1-c}\Bigg(1 &+ \frac{(a-c+1)(b-c+1)}{1!\,(-c+2)}x \\ &+ \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!\,(-c+2)(-c+3)}x^2 \\ &+ \cdots\Bigg). \end{aligned}$$

Show that

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

(e) On the generality of the hypergeometric equation. Show that

$$(18)\quad (t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0$$

with $\dot{y} = dy/dt$, etc., constant A, B, C, D, K, and $t^2 + At + B = (t - t_1)(t - t_2)$, $t_1 \neq t_2$, can be reduced to the hypergeometric equation with independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

and parameters related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$. From this you see that (15) is a "normalized form" of the more general (18) and that various cases of (18) can thus be solved in terms of hypergeometric functions.

19–24 HYPERGEOMETRIC EQUATIONS

Find a general solution in terms of hypergeometric functions.

19. $x(1-x)y'' + (\frac{1}{2} - 2x)y' - \frac{1}{4}y = 0$

20. $2x(1-x)y'' - (1+6x)y' - 2y = 0$

21. $x(1-x)y'' + \frac{1}{2}y' + 2y = 0$

22. $3t(1+t)\ddot{y} + t\dot{y} - y = 0$

23. $2(t^2 - 5t + 6)\ddot{y} + (2t - 3)\dot{y} - 8y = 0$

24. $4(t^2 - 3t + 2)\ddot{y} - 2\dot{y} + y = 0$

[5] CARL FRIEDRICH GAUSS (1777–1855), great German mathematician. He already made the first of his great discoveries as a student at Helmstedt and Göttingen. In 1807 he became a professor and director of the Observatory at Göttingen. His work was of basic importance in algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, numeric analysis, astronomy, geodesy, electromagnetism, and theoretical mechanics. He also paved the way for a general and systematic use of complex numbers.

5.5 Bessel's Equation. Bessel Functions $J_\nu(x)$

One of the most important ODEs in applied mathematics in **Bessel's equation**,[6]

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{1}$$

Its diverse applications range from electric fields to heat conduction and vibrations (see Sec. 12.9). It often appears when a problem shows *cylindrical symmetry* (just as Legendre's equation may appear in cases of *spherical* symmetry). The parameter ν in (1) is a given number. We assume that ν is real and nonnegative.

Bessel's equation can be solved by the Frobenius method, as we mentioned at the beginning of the preceding section, where the equation is written in standard form (obtained by dividing (1) by x^2). Accordingly, we substitute the series

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{2}$$

with undetermined coefficients and its derivatives into (1). This gives

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero. Note that this power x^{s+r} corresponds to $m = s$ in the first, second, and fourth series, and to $m = s - 2$ in the third series. Hence for $s = 0$ and $s = 1$, the third series does not contribute since $m \geqq 0$. For $s = 2, 3, \cdots$ all four series contribute, so that we get a general formula for all these s. We find

$$\begin{aligned}
&\text{(a)} \quad r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 && (s = 0)\\
\text{(3)}\quad &\text{(b)} \quad (r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 && (s = 1)\\
&\text{(c)} \quad (s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 && (s = 2, 3, \cdots).
\end{aligned}$$

From (3a) we obtain the **indicial equation** by dropping a_0,

$$(r+\nu)(r-\nu) = 0. \tag{4}$$

The roots are $r_1 = \nu\ (\geqq 0)$ and $r_2 = -\nu$.

---

[6] FRIEDRICH WILHELM BESSEL (1784–1846), German astronomer and mathematician, studied astronomy on his own in his spare time as an apprentice of a trade company and finally became director of the new Königsberg Observatory.

Formulas on Bessel functions are contained in Ref. [GR1] and the standard treatise [A13].

Coefficient Recursion for $r = r_1 = \nu$. For $r = \nu$, Eq. (3b) reduces to $(2\nu + 1)a_1 = 0$. Hence $a_1 = 0$ since $\nu \geqq 0$. Substituting $r = \nu$ in (3c) and combining the three terms containing a_s gives simply

(5) $$(s + 2\nu)sa_s + a_{s-2} = 0.$$

Since $a_1 = 0$ and $\nu \geqq 0$, it follows from (5) that $a_3 = 0$, $a_5 = 0, \cdots$. Hence we have to deal only with *even-numbered* coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (5) becomes

$$(2m + 2\nu)2ma_{2m} + a_{2m-2} = 0.$$

Solving for a_{2m} gives the recursion formula

(6) $$a_{2m} = -\frac{1}{2^2 m(\nu + m)} a_{2m-2}, \qquad m = 1, 2, \cdots.$$

From (6) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(\nu + 1)}$$

$$a_4 = -\frac{a_2}{2^2 2(\nu + 2)} = \frac{a_0}{2^4 2!\,(\nu + 1)(\nu + 2)}$$

and so on, and in general

(7) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(\nu + 1)(\nu + 2) \cdots (\nu + m)}, \qquad m = 1, 2, \cdots.$$

Bessel Functions $J_n(x)$ For Integer $\nu = n$

Integer values of v are denoted by n. This is standard. For $\nu = n$ the relation (7) becomes

(8) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n + 1)(n + 2) \cdots (n + m)}, \qquad m = 1, 2, \cdots.$$

a_0 is still arbitrary, so that the series (2) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly, we have to make a choice. $a_0 = 1$ would be possible, but more practical turns out to be

(9) $$a_0 = \frac{1}{2^n n!}.$$

because then $n!(n + 1) \cdots (n + m) = (m + n)!$ in (8), so that (8) simply becomes

(10) $$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n + m)!}, \qquad m = 1, 2, \cdots.$$

This simplicity of the denominator of (10) partially motivates the choice (9). With these coefficients and $r_1 = \nu = n$ we get from (2) a particular solution of (1), denoted by $J_n(x)$ and given by

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}\,. \tag{11}$$

$J_n(x)$ is called the **Bessel function of the first kind** *of order n.* The series (11) converges for all x, as the ratio test shows. In fact, it converges very rapidly because of the factorials in the denominator.

EXAMPLE 1 Bessel Functions $J_0(x)$ and $J_1(x)$

For $n = 0$ we obtain from (11) the **Bessel function of order 0**

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots \tag{12}$$

which looks similar to a cosine (Fig. 107). For $n = 1$ we obtain the **Bessel function of order 1**

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + - \cdots, \tag{13}$$

which looks similar to a sine (Fig. 107). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing x. Heuristically, n^2/x^2 in (1) in standard form [(1) divided by x^2] is zero (if $n = 0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y'' + y = 0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{14}$$

where $\sim$ is read **"asymptotically equal"** and means that *for fixed n* the quotient of the two sides approaches 1 as $x \to \infty$.

Formula (14) is surprisingly accurate even for smaller x (> 0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of J_0 you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc. ■

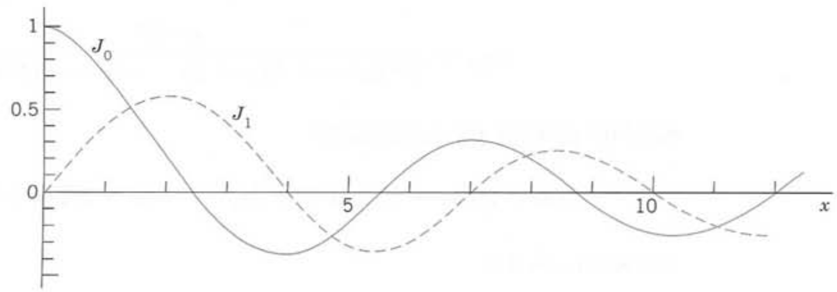


Fig. 107. Bessel functions of the first kind J_0 and J_1

Bessel Functions $J_\nu(x)$ for any $\nu \geqq 0$. Gamma Function

We now extend our discussion from integer $\nu = n$ to any $\nu \geqq 0$. All we need is an extension of the factorials in (9) and (11) to any ν. This is done by the **gamma function** $\Gamma(\nu)$ defined by the integral

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\, dt \qquad (\nu > 0). \tag{15}$$

By integration by parts we obtain

$$\Gamma(\nu + 1) = \int_0^\infty e^{-t} t^\nu\, dt = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1}\, dt.$$

The first expression on the right is zero. The integral on the right is $\Gamma(\nu)$. This yields the basic functional relation

$$\Gamma(\nu + 1) = \nu\,\Gamma(\nu). \tag{16}$$

Now by (15)

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t} \Big|_0^\infty = 0 - (-1) = 1.$$

From this and (16) we obtain successively $\Gamma(2) = \Gamma(1) = 1!$, $\Gamma(3) = 2\Gamma(2) = 2!$, $\cdots$ and in general

$$\Gamma(n + 1) = n! \qquad (n = 0, 1, \cdots). \tag{17}$$

This shows the ***the gamma function does in fact generalize the factorial function.***

Now in (9) we had $a_0 = 1/(2^n n!)$. This is $1/(2^n \Gamma(n + 1))$ by (17). It suggests to choose, for any ν,

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}. \tag{18}$$

Then (7) becomes

$$a_{2m} = \frac{(-1)^m}{2^{2m} m!\, (\nu + 1)(\nu + 2) \cdots (\nu + m) 2^\nu \Gamma(\nu + 1)}.$$

But (16) gives in the denominator

$$(\nu + 1)\Gamma(\nu + 1) = \Gamma(\nu + 2), \qquad (\nu + 2)\Gamma(\nu + 2) = \Gamma(\nu + 3)$$

and so on, so that

$$(\nu + 1)(\nu + 2) \cdots (\nu + m)\Gamma(\nu + 1) = \Gamma(\nu + m + 1).$$

Hence because of our (standard!) choice (18) of a_0 the coefficients (7) simply are

$$a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)}. \tag{19}$$

With these coefficients and $r = r_1 = \nu$ we get from (2) a particular solution of (1), denoted by $J_\nu(x)$ and given by

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)}. \tag{20}$$

$J_\nu(x)$ is called the **Bessel function of the first kind of order ν.** The series (20) converges for all x, as one can verify by the ratio test.

General Solution for Noninteger ν. Solution $J_{-\nu}$

For a general solution, in addition to J_ν we need a second linearly independent solution. For ν not an integer this is easy. Replacing ν by $-\nu$ in (20), we have

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\,\Gamma(m - \nu + 1)}. \tag{21}$$

Since Bessel's equation involves ν^2, the functions J_ν and $J_{-\nu}$ are solutions of the equation for the same ν. If ν is not an integer, they are linearly independent, because the first term in (20) and the first term in (21) are finite nonzero multiples of x^ν and $x^{-\nu}$, respectively. $x = 0$ must be excluded in (21) because of the factor $x^{-\nu}$ (with $\nu > 0$). This gives

THEOREM 1

General Solution of Bessel's Equation

If ν is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x). \tag{22}$$

But if ν is an integer, then (22) is not a general solution because of linear dependence:

THEOREM 2

Linear Dependence of Bessel Functions J_n and J_{-n}

For integer $\nu = n$ the Bessel functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots). \tag{23}$$

PROOF We use (21) and let ν approach a positive integer n. Then the gamma functions in the coefficients of the first n terms become infinite (see Fig. 552 in App. A3.1), the coefficients become zero, and the summation starts with $m = n$. Since in this case $\Gamma(m - n + 1) = (m - n)!$ by (17), we obtain

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!\,(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!\,s!} \qquad (m = n + s).$$

The last series represents $(-1)^n J_n(x)$, as you can see from (11) with m replaced by s. This completes the proof. ■

A general solution for integer n will be given in the next section, based on some further interesting ideas.

Discovery of Properties From Series

Bessel functions are a model case for showing how to discover properties and relations of functions from series by which they are *defined.* Bessel functions satisfy an incredibly large number of relationships—look at Ref. [A13] in App. 1; also, find out what your CAS knows. In Theorem 3 we shall discuss four formulas that are backbones in applications.

THEOREM 3 **Derivatives, Recursions**

The derivative of $J_\nu(x)$ with respect to x can be expressed by $J_{\nu-1}(x)$ or $J_{\nu+1}(x)$ by the formulas

$$\textbf{(24)} \qquad \begin{aligned} &\textbf{(a)} \quad [x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x) \\ &\textbf{(b)} \quad [x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x). \end{aligned}$$

Furthermore, $J_\nu(x)$ and its derivative satisfy the recurrence relations

$$\textbf{(24)} \qquad \begin{aligned} &\textbf{(c)} \quad J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \\ &\textbf{(d)} \quad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x). \end{aligned}$$

PROOF **(a)** We multiply (20) by x^ν and take $x^{2\nu}$ under the summation sign. Then we have

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)}.$$

We now differentiate this, cancel a factor 2, pull $x^{2\nu-1}$ out, and use the functional relationship $\Gamma(\nu + m + 1) = (\nu + m)\Gamma(\nu + m)$ [see (16)]. Then (20) with $\nu - 1$ instead of ν shows that we obtain the right side of (24a). Indeed,

$$(x^\nu J_\nu)' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu) x^{2m+2\nu-1}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu-1} m!\,\Gamma(\nu + m)}.$$

(b) Similarly, we multiply (20) by $x^{-\nu}$, so that x^ν in (20) cancels. Then we differentiate, cancel $2m$, and use $m! = m(m-1)!$. This gives, with $m = s + 1$,

$$(x^{-\nu}J_\nu)' = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+\nu-1}(m-1)!\,\Gamma(\nu+m+1)} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\nu+1} s!\,\Gamma(\nu+s+2)}\,.$$

Equation (20) with $\nu + 1$ instead of ν and s instead of m shows that the expression on the right is $-x^{-\nu}J_{\nu+1}(x)$. This proves (24b).

(c), (d) We perform the differentiation in (24a). Then we do the same in (24b) and multiply the result on both sides by $x^{2\nu}$. This gives

$$\text{(a*)} \qquad \nu x^{\nu-1}J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1}$$

$$\text{(b*)} \qquad -\nu x^{\nu-1}J_\nu + x^\nu J_\nu' = -x^\nu J_{\nu+1}.$$

Substracting (b*) from (a*) and dividing the result by x^ν gives (24c). Adding (a*) and (b*) and dividing the result by x^ν gives (24d). ■

EXAMPLE 2 Application of Theorem 3 in Evaluation and Integration

Formula (24c) can be used recursively in the form

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$$

for calculating Bessel functions of higher order from those of lower order. For instance, $J_2(x) = 2J_1(x)/x - J_0(x)$, so that J_2 can be obtained from tables of J_0 and J_1 (in App. 5 or, more accurately, in Ref. [GR1] in App. 1).

To illustrate how Theorem 3 helps in integration, we use (24b) with $\nu = 3$ integrated on both sides. This evaluates, for instance, the integral

$$I = \int_1^2 x^{-3}J_4(x)\,dx = -x^{-3}J_3(x)\Big|_1^2 = -\frac{1}{8}J_3(2) + J_3(1).$$

A table of J_3 (on p. 398 of Ref. [GR1]) or your CAS will give you

$$-\tfrac{1}{8}\cdot 0.128943 + 0.019563 = 0.003445.$$

Your CAS (or a human computer in precomputer times) obtains J_3 from (24), first using (24c) with $\nu = 2$, that is, $J_3 = 4x^{-1}J_2 - J_1$, then (24c) with $\nu = 1$, that is, $J_2 = 2x^{-1}J_1 - J_0$. Together,

$$\begin{aligned} I &= x^{-3}(4x^{-1}(2x^{-1}J_1 - J_0) - J_1)\Big|_1^2 \\ &= -\tfrac{1}{8}[2J_1(2) - 2J_0(2) - J_1(2)] + [8J_1(1) - 4J_0(1) - J_1(1)] \\ &= -\tfrac{1}{8}J_1(2) + \tfrac{1}{4}J_0(2) + 7J_1(1) - 4J_0(1). \end{aligned}$$

This is what you get, for instance, with Maple if you type int(· · ·). And if you type evalf(int(· · ·)), you obtain 0.003445448, in agreement with the result near the beginning of the example. ■

In the theory of special functions it often happens that for certain values of a parameter a higher function becomes elementary. We have seen this in the last problem set, and we now show this for J_ν.

THEOREM 4

Elementary J_ν for Half-Integer Order ν

Bessel functions J_ν of orders $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ are elementary; they can be expressed by finitely many cosines and sines and powers of x. In particular,

$$\text{(25)} \qquad \text{(a)}\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x, \qquad \text{(b)}\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x.$$

PROOF When $\nu = \frac{1}{2}$, then (20) is

$$J_{1/2}(x) = \sqrt{x}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+1/2} m!\,\Gamma(m+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,\Gamma(m+\frac{3}{2})}.$$

To simplify the denominator, we first write it out as a product AB, where

$$A = 2^m m! = 2m(2m-2)(2m-4)\cdots 4\cdot 2$$

and [use (16)]

$$B = 2^{m+1}\Gamma(m+\tfrac{3}{2}) = 2^{m+1}(m+\tfrac{1}{2})(m-\tfrac{1}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$
$$= (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi};$$

here we used

$$\text{(26)} \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

We see that the product of the two right sides of A and B is simply $(2m+1)!\sqrt{\pi}$, so that $J_{1/2}$ becomes

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\,\sin x,$$

as claimed. Differentiation and the use of (24a) with $\nu = \frac{1}{2}$ now gives

$$[\sqrt{x}J_{1/2}(x)]' = \sqrt{\frac{2}{\pi}}\,\cos x = x^{1/2}J_{-1/2}(x).$$

This proves (25b). From (25) follow further formulas successively by (24c), used as in Example 2. This completes the proof. ■

EXAMPLE 3 **Further Elementary Bessel Functions**

From (24c) with $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$ and (25) we obtain

$$J_{3/2}(x) = \frac{1}{x}\,J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-3/2}(x) = -\frac{1}{x}\,J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

respectively, and so on. ■

We hope that our study has not only helped you to become acquainted with Bessel functions but has also convinced you that series can be quite useful in obtaining various properties of the corresponding functions.

PROBLEM SET 5.5

1. **(Convergence)** Show that the series in (11) converges for all x. Why is the convergence very rapid?
2. **(Approximation)** Show that for small $|x|$ we have $J_0 \approx 1 - 0.25x^2$. From this compute $J_0(x)$ for $x = 0, 0.1, 0.2, \cdots, 1.0$ and determine the error by using Table A1 in App. 5 or your CAS.
3. **("Large" values)** Using (14), compute $J_0(x)$ for $x = 1.0, 2.0, 3.0, \cdots, 8.0$, determine the error by Table A1 or your CAS, and comment.
4. **(Zeros)** Compute the first four positive zeros of $J_0(x)$ and $J_1(x)$ from (14). Determine the error and comment.

5–20 ODEs REDUCIBLE TO BESSEL'S EQUATION

Using the indicated substitutions, find a general solution in terms of J_ν and $J_{-\nu}$ or indicate when this is not possible. (This is just a sample of various ODEs reducible to Bessel's equation. Some more follow in the next problem set. Show the details of your work.)

5. **(ODE with two parameters)** $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad (\lambda x = z)$
6. $x^2y'' + xy' + (x^2 - \frac{1}{16})y = 0$
7. $x^2y'' + xy' + \frac{1}{4}(x - \nu^2)y = 0 \quad (\sqrt{x} = z)$
8. $(2x + 1)^2y'' + 2(2x + 1)y' + 16x(x + 1)y = 0 \quad (2x + 1 = z)$
9. $xy'' - y' + 4xy = 0 \quad (y = xu,\ 2x = z)$
10. $x^2y'' + xy' + \frac{1}{4}(x^2 - 1)y = 0 \quad (x = 2z)$
11. $xy'' + (2\nu + 1)y' + xy = 0 \quad (y = x^{-\nu}u)$
12. $x^2y'' + xy' + 4(x^4 - \nu^2)y = 0 \quad (x^2 = z)$
13. $x^2y'' + xy' + 9(x^6 - \nu^2)y = 0 \quad (x^3 = z)$
14. $y'' + (e^{2x} - \frac{1}{9})y = 0 \quad (e^x = z)$
15. $xy'' + y = 0 \quad (y = \sqrt{x}\,u,\ 2\sqrt{x} = z)$
16. $16x^2y'' + 8xy' + (x^{1/2} + \frac{15}{16})y = 0 \quad (y = x^{1/4}u,\ x^{1/4} = z)$
17. $36x^2y'' + 18xy' + \sqrt{x}\,y = 0 \quad (y = x^{1/4}u,\ \frac{2}{3}x^{1/4} = z)$
18. $x^2y'' + xy' + \sqrt{x}\,y = 0 \quad (4x^{1/4} = z)$
19. $x^2y'' + \frac{1}{5}xy' + \sqrt{x}\,y = 0 \quad (y = x^{2/5}u,\ 4x^{1/4} = z)$
20. $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0 \quad (y = x^\nu u,\ x^\nu = z)$

21–28 APPLICATION OF (24): DERIVATIVES, INTEGRALS

Use the powerful formulas (24) to do Probs. 21–28. (Show the details of your work.)

21. **(Derivatives)** Show that $J_0'(x) = -J_1(x)$, $J_1'(x) = J_0(x) - J_1(x)/x$, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$.
22. **(Interlacing of zeros)** Using (24) and Rolle's theorem, show that between two consecutive zeros of $J_0(x)$ there is precisely one zero of $J_1(x)$.
23. **(Interlacing of zeros)** Using (24) and Rolle's theorem, show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.
24. **(Bessel's equation)** Derive (1) from (24).
25. **(Basic integral formulas)** Show that

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c,$$

$$\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_\nu(x) + c,$$

$$\int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x).$$

26. **(Integration)** Evaluate $\int x^{-1}J_4(x)\,dx$. (Use Prob. 25; integrate by parts.)
27. **(Integration)** Show that $\int x^2J_0(x)\,dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)\,dx$. (The last integral is nonelementary; tables exist, e.g. in Ref. [A13] in App. 1.)
28. **(Integration)** Evaluate $\int J_5(x)\,dx$.
29. **(Elimination of first derivative)** Show that $y = uv$ with $v(x) = \exp(-\frac{1}{2}\int p(x)\,dx)$ gives from the ODE $y'' + p(x)y' + q(x)y = 0$ the ODE

$$u'' + \left[q(x) - \tfrac{1}{4}p(x)^2 - \tfrac{1}{2}p'(x)\right]u = 0$$

no longer containing the first derivative. Show that for the Bessel equation the substitution is $y = ux^{-1/2}$ and gives

$$x^2u'' + (x^2 + \tfrac{1}{4} - \nu^2)u = 0. \tag{27}$$

30. (Elementary Bessel functions) Derive (25) in Theorem 4 from (27).

31. CAS EXPERIMENT. Change of Coefficient. Find and graph (on common axes) the solutions of

$$y'' + kx^{-1}y' + y = 0,\ y(0) = 1,\ y'(0) = 0,$$

for $k = 0, 1, 2, \cdots, 10$ (or as far as you get useful graphs). For what k do you get elementary functions? Why? Try for noninteger k, particularly between 0 and 2, to see the continuous change of the curve. Describe the change of the location of the zeros and of the extrema as k increases from 0. Can you interpret the ODE as a model in mechanics, thereby explaining your observations?

32. TEAM PROJECT. Modeling a Vibrating Cable (Fig. 108). A flexible cable, chain, or rope of length L and density (mass per unit length) ρ is fixed at the upper end ($x = 0$) and allowed to make small vibrations (small angles α in the horizontal displacement $u(x, t)$, t = time) in a vertical plane.

(a) Show the following. The weight of the cable below a point x is $W(x) = \rho g(L - x)$. The restoring force is $F(x) = W \sin \alpha \approx Wu_x$, $u_x = \partial u/\partial x$. The difference in force between x and $x + \Delta x$ is $\Delta x\, (Wu_x)_x$. Newton's second law now gives

$$\rho\, \Delta x\, u_{tt} = \Delta x\, \rho g[(L - x)u_x]_x.$$

For the expected periodic motion $u(x, t) = y(x) \cos(\omega t + \delta)$ the model of the problem is the ODE

$$(L - x)y'' - y' + \lambda^2 y = 0, \qquad \lambda^2 = \omega^2/g.$$

(b) Transform this ODE to $\ddot{y} + s^{-1}\dot{y} + y = 0$, $\dot{y} = dy/ds$, $s = 2\lambda z^{1/2}$, $z = L - x$, so that the solution is

$$y(x) = J_0(2\omega\sqrt{(L - x)/g}).$$

(c) Conclude that possible frequencies $\omega/2\pi$ are those for which $s = 2\omega\sqrt{L/g}$ is a zero of J_0. The corresponding solutions are called **normal modes.** Figure 108 shows the first of them. What does the second normal mode look like? The third? What is the frequency (cycles/min) of a cable of length 2 m? Of length 10 m?

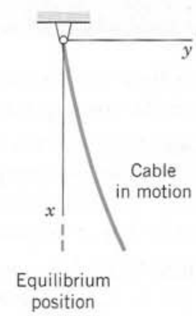


Fig. 108. Vibrating cable in Team Project 32

33. CAS EXPERIMENT. Bessel Functions for Large x.

(a) Graph $J_n(x)$ for $n = 0, \cdots, 5$ on common axes.

(b) Experiment with (14) for integer n. Using graphs, find out from which $x = x_n$ on the curves of (11) and (14) practically coincide. How does x_n change with n?

(c) What happens in (b) if $n = \pm\frac{1}{2}$? (Our usual notation in this case would be ν.)

(d) How does the error of (14) behave as function of x for fixed n? [Error = exact value minus approximation (14).]

(e) Show from the graphs that $J_0(x)$ has extrema where $J_1(x) = 0$. Which formula proves this? Find further relations between zeros and extrema.

(f) Raise and answer questions of your own, for instance, on the zeros of J_0 and J_1. How accurately are they obtained from (14)?

5.6 Bessel Functions of the Second Kind $Y_\nu(x)$

From the last section we know that J_ν and $J_{-\nu}$ form a basis of solutions of Bessel's equation, provided ν is not an integer. But when ν is an integer, these two solutions are linearly dependent on any interval (see Theorem 2 in Sec. 5.5). Hence to have a general solution also when $\nu = n$ is an integer, we need a second linearly independent solution besides J_n. This solution is called a **Bessel function of the second kind** and is denoted by Y_n. We shall now derive such a solution, beginning with the case $n = 0$.

$n = 0$: Bessel Function of the Second Kind $Y_0(x)$

When $n = 0$, Bessel's equation can be written

$$xy'' + y' + xy = 0. \tag{1}$$

Then the indicial equation (4) in Sec. 5.5 has a double root $r = 0$. This is Case 2 in Sec. 5.4. In this case we first have only one solution, $J_0(x)$. From (8) in Sec. 5.4 we see that the desired second solution must be of the form

$$(2) \qquad y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m.$$

We substitute y_2 and its derivatives

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

into (1). Then the sum of the three logarithmic terms $xJ_0'' \ln x$, $J_0' \ln x$, and $xJ_0 \ln x$ is zero because J_0 is a solution of (1). The terms $-J_0/x$ and J_0/x (from xy'' and y') cancel. Hence we are left with

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

Addition of the first and second series gives $\Sigma m^2 A_m x^{m-1}$. The power series of $J_0'(x)$ is obtained from (12) in Sec. 5.5 and the use of $m!/m = (m-1)!$ in the form

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!\,(m-1)!}.$$

Together with $\Sigma m^2 A_m x^{m-1}$ and $\Sigma A_m x^{m+1}$ this gives

$$(3^*) \qquad \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!\,(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

First, we show that the A_m with odd subscripts are all zero. The power x^0 occurs only in the second series, with coefficient A_1. Hence $A_1 = 0$. Next, we consider the even powers x^{2s}. The first series contains none. In the second series, $m - 1 = 2s$ gives the term $(2s+1)^2 A_{2s+1} x^{2s}$. In the third series, $m + 1 = 2s$. Hence by equating the sum of the coefficients of x^{2s} to zero we have

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 1, 2, \cdots.$$

Since $A_1 = 0$, we thus obtain $A_3 = 0$, $A_5 = 0$, $\cdots$, successively.

We now equate the sum of the coefficients of x^{2s+1} to zero. For $s = 0$ this gives

$$-1 + 4A_2 = 0, \qquad \text{thus} \qquad A_2 = \tfrac{1}{4}.$$

For the other values of s we have in the first series in (3*) $2m - 1 = 2s + 1$, hence $m = s + 1$, in the second $m - 1 = 2s + 1$, and in the third $m + 1 = 2s + 1$. We thus obtain

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0.$$

For $s = 1$ this yields

$$\frac{1}{8} + 16A_4 + A_2 = 0, \qquad \text{thus} \qquad A_4 = -\frac{3}{128}$$

and in general

$$(3) \qquad A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \qquad m = 1, 2, \cdots.$$

Using the short notations

$$(4) \qquad h_1 = 1 \qquad h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m} \qquad m = 2, 3, \cdots$$

and inserting (4) and $A_1 = A_3 = \cdots = 0$ into (2), we obtain the result

$$(5) \qquad \begin{aligned} y_2(x) &= J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}\, x^{2m} \\ &= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - + \cdots. \end{aligned}$$

Since J_0 and y_2 are linearly independent functions, they form a basis of (1) for $x > 0$. Of course, another basis is obtained if we replace y_2 by an independent particular solution of the form $a(y_2 + bJ_0)$, where a ($\neq 0$) and b are constants. It is customary to choose $a = 2/\pi$ and $b = \gamma - \ln 2$, where the number $\gamma = 0.577\,215\,664\,90 \cdots$ is the so-called **Euler constant,** which is defined as the limit of

$$1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

as s approaches infinity. The standard particular solution thus obtained is called the **Bessel function of the second kind** *of order zero* (Fig. 109) or **Neumann's function** *of order zero* and is denoted by $Y_0(x)$. Thus [see (4)]

$$\textbf{(6)} \qquad Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}\, x^{2m}\right].$$

For small $x > 0$ the function $Y_0(x)$ behaves about like $\ln x$ (see Fig. 109, why?), and $Y_0(x) \to -\infty$ as $x \to 0$.

Bessel Functions of the Second Kind $Y_n(x)$

For $\nu = n = 1, 2, \cdots$ a second solution can be obtained by manipulations similar to those for $n = 0$, starting from (10), Sec 5.4. It turns out that in these cases the solution also contains a logarithmic term.

The situation is not yet completely satisfactory, because the second solution is defined differently, depending on whether the order ν is an integer or not. To provide uniformity of formalism, it is desirable to adopt a form of the second solution that is valid for all values of the order. For this reason we introduce a standard second solution $Y_\nu(x)$ defined for all ν by the formula

$$\text{(7)}\quad \begin{aligned} &\text{(a)} && Y_\nu(x) = \frac{1}{\sin \nu\pi}\,[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)] \\ &\text{(b)} && Y_n(x) = \lim_{\nu\to n} Y_\nu(x). \end{aligned}$$

This function is called the **Bessel function of the second kind** *of order* ν or **Neumann's function**[7] *of order* ν. Figure 109 shows $Y_0(x)$ and $Y_1(x)$.

Let us show that J_ν and Y_ν are indeed linearly independent for all ν (and $x > 0$).

For noninteger order ν, the function $Y_\nu(x)$ is evidently a solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are solutions of that equation. Since for those ν the solutions J_ν and $J_{-\nu}$ are linearly independent and Y_ν involves $J_{-\nu}$, the functions J_ν and Y_ν are linearly independent. Furthermore, it can be shown that the limit in (7b) exists and Y_n is a solution of Bessel's equation for integer order; see Ref. [A13] in App. 1. We shall see that the series development of $Y_n(x)$ contains a logarithmic term. Hence $J_n(x)$ and $Y_n(x)$ are linearly independent solutions of Bessel's equation. The series development of $Y_n(x)$ can be obtained if we insert the series (20) and (21), Sec. 5.5, for $J_\nu(x)$ and $J_{-\nu}(x)$ into (7a) and then let ν approach n; for details see Ref. [A13]. The result is

$$\text{(8)}\quad \begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) &+ \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!\,(m+n)!}\,x^{2m} \\ &- \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n} m!}\,x^{2m} \end{aligned}$$

where $x > 0$, $n = 0, 1, \cdots$, and [as in (4)] $h_0 = 0$, $h_1 = 1$,

$$h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \qquad h_{m+n} = 1 + \frac{1}{2} + \cdots + \frac{1}{m+n}.$$

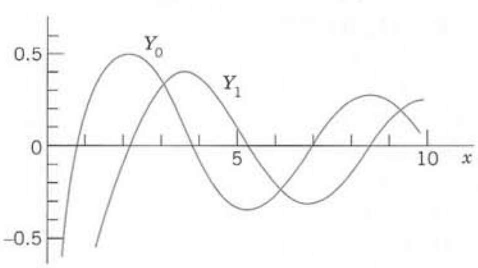


Fig. 109. Bessel functions of the second kind Y_0 and Y_1. (For a small table, see App. 5.)

---

[7] CARL NEUMANN (1832–1925), German mathematician and physicist. His work on potential theory sparked the development in the field of integral equations by VITO VOLTERRA (1860–1940) of Rome, ERIC IVAR FREDHOLM (1866–1927) of Stockholm, and DAVID HILBERT (1862–1943) of Göttingen (see the footnote in Sec. 7.9).

The solutions $Y_\nu(x)$ are sometimes denoted by $N_\nu(x)$; in Ref. [A13] they are called **Weber's functions;** Euler's constant in (6) is often denoted by C or $\ln \gamma$.

For $n = 0$ the last sum in (8) is to be replaced by 0 [giving agreement with (6)].
Furthermore, it can be shown that

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

Our main result may now be formulated as follows.

THEOREM 1

General Solution of Bessel's Equation

A general solution of Bessel's equation for all values of ν *(and* $x > 0$*) is*

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \quad (9)$$

We finally mention that there is a practical need for solutions of Bessel's equation that are complex for real values of x. For this purpose the solutions

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \end{aligned} \quad (10)$$

are frequently used. These linearly independent functions are called **Bessel functions of the third kind** *of order* ν or *first and second* **Hankel functions**[8] *of order* ν.

This finishes our discussion on Bessel functions, except for their "orthogonality," which we explain in Sec. 5.7. Applications to vibrations follow in Sec. 12.9.

PROBLEM SET 5.6

1–10 SOME FURTHER ODEs REDUCIBLE TO BESSEL'S EQUATIONS

(See also Sec. 5.5.)

Using the indicated substitutions, find a general solution in terms of J_ν and Y_ν. Indicate whether you could also use $J_{-\nu}$ instead of Y_ν. (Show the details of your work.)

1. $x^2y'' + xy' + (x^2 - 25)y = 0$
2. $x^2y'' + xy' + (9x^2 - \frac{1}{9})y = 0 \quad (3x = z)$
3. $4xy'' + 4y' + y = 0 \quad (\sqrt{x} = z)$
4. $xy'' + y' + 36y = 0 \quad (12\sqrt{x} = z)$
5. $x^2y'' + xy' + (4x^4 - 16)y = 0 \quad (x^2 = z)$
6. $x^2y'' + xy' + (x^6 - 1)y = 0 \quad (\frac{1}{3}x^3 = z)$
7. $xy'' + 11y' + xy = 0 \quad (y = x^{-5}u)$
8. $y'' + 4x^2y = 0 \quad (y = u\sqrt{x}, x^2 = z)$
9. $x^2y'' - 5xy' + 9(x^6 - 8)y = 0 \quad (y = x^3u, x^3 = z)$
10. $xy'' + 7y' + 4xy = 0 \quad (y = x^{-3}u, 2x = z)$
11. **(Hankel functions)** Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any ν.
12. **CAS EXPERIMENT. Bessel Functions for Large x.** It can be shown that for large x,

$$Y_n(x) \sim \sqrt{2/(\pi x)} \sin\left(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right) \quad (11)$$

with $\sim$ defined as in (14) of Sec. 5.5.

(a) Graph $Y_n(x)$ for $n = 0, \cdots, 5$ on common axes. Are there relations between zeros of one function and extrema of another? For what functions?

(b) Find out from graphs from which $x = x_n$ on the curves of (8) and (11) (both obtained from your CAS) practically coincide. How does x_n change with n?

(c) Calculate the first ten zeros x_m, $m = 1, \cdots, 10$, of $Y_0(x)$ from your CAS and from (11). How does the error behave as m increases?

(d) Do (c) for $Y_1(x)$ and $Y_2(x)$. How do the errors compare to those in (c)?

[8] HERMANN HANKEL (1839–1873), German mathematician.

13. Modified Bessel functions of the first kind of order ν are defined by $I_\nu(x) = i^{-\nu} J_\nu(ix)$, $i = \sqrt{-1}$. Show that I_ν satisfies the ODE

(12) $$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

and has the representation

(13) $$I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu} m!\, \Gamma(m + \nu + 1)}.$$

14. (Modified Bessel functions I_ν) Show that $I_\nu(x)$ is real for all real x (and real ν), $I_\nu(x) \neq 0$ for all real $x \neq 0$, and $I_{-n}(x) = I_n(x)$, where n is any integer.

15. Modified Bessel functions of the third kind (sometimes called *of the second kind*) are defined by the formula (14) below. Show that they satisfy the ODE (12).

(14) $$K_\nu(x) = \frac{\pi}{2 \sin \nu\pi} [I_{-\nu}(x) - I_\nu(x)]$$

5.7 Sturm–Liouville Problems. Orthogonal Functions

So far we have considered initial value problems. We recall from Sec. 2.1 that such a problem consists of an ODE, say, of second order, and initial conditions $y(x_0) = K_0$, $y'(x_0) = K_1$ referring to the *same point* (initial point) $x = x_0$. We now turn to boundary value problems. A **boundary value problem** consists of an ODE and given **boundary conditions** referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leqq x \leqq b$. To solve such a problem means to find a solution of the ODE on the interval $a \leqq x \leqq b$ satisfying the boundary conditions.

We shall see that Legendre's, Bessel's, and other ODEs of importance in engineering can be written as a **Sturm–Liouville equation**

(1) $$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

involving a parameter λ. The boundary value problem consisting of an ODE (1) and given **Sturm–Liouville boundary conditions**

(2)
$$\text{(a)}\quad k_1 y(a) + k_2 y'(a) = 0$$
$$\text{(b)}\quad l_1 y(b) + l_2 y'(b) = 0$$

is called a **Sturm–Liouville problem.**[9] We shall see further that these problems lead to useful series developments in terms of particular solutions of (1), (2). Crucial in this connection is *orthogonality* to be discussed later in this section.

In (1) we make the ***assumptions*** that p, q, r, and p' are continuous on $a \leqq x \leqq b$, and

$$r(x) > 0 \qquad (a \leqq x \leqq b).$$

In (2) we assume that k_1, k_2 are given constants, not both zero, and so are l_1, l_2, not both zero.

---

[9]JACQUES CHARLES FRANÇOIS STURM (1803–1855), was born and studied in Switzerland and then moved to Paris, where he later became the successor of Poisson in the chair of mechanics at the Sorbonne (the University of Paris).

JOSEPH LIOUVILLE (1809–1882), French mathematician and professor in Paris, contributed to various fields in mathematics and is particularly known by his important work in complex analysis (Liouville's theorem; Sec. 14.4), special functions, differential geometry, and number theory.

EXAMPLE 1 Legendre's and Bessel's Equations are Sturm–Liouville Equations

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \qquad \lambda = n(n + 1).$$

This is (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$.

In Bessel's equation

$$\tilde{x}^2\ddot{y} + \tilde{x}\dot{y} + (\tilde{x}^2 - n^2)y = 0 \qquad \dot{y} = dy/d\tilde{x}, \text{ etc.}$$

as a model in physics or elsewhere, one often likes to have another parameter k in addition to n. For this reason we set $\tilde{x} = kx$. Then by the chain rule $\dot{y} = dy/d\tilde{x} = (dy/dx)\, dx/d\tilde{x} = y'/k$, $\ddot{y} = y''/k^2$. In the first two terms, k^2 and k drop out and we get

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0.$$

Division by x gives the Sturm–Liouville equation

$$[xy']' + \left(-\frac{n^2}{x} + \lambda x\right)y = 0 \qquad \lambda = k^2.$$

This is (1) with $p = x$, $q = -n^2/x$, and $r = x$. ■

Eigenfunctions, Eigenvalues

Clearly, $y \equiv 0$ is a solution—the **"trivial solution"**—for any λ because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find **eigenfunctions** $y(x)$, that is, solutions of (1) satisfying (2) without being identically zero. We call a number λ for which an eigenfunction exists an **eigenvalue** of the Sturm–Liouville problem (1), (2).

EXAMPLE 2 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$(3) \qquad y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0.$$

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and then fixed at its ends $x = 0$ and $x = \pi$ and allowed to vibrate. Then $y(x)$ is the "space function" of the deflection $u(x, t)$ of the string, assumed in the form $u(x, t) = y(x)w(t)$, where t is time. (This model will be discussed in great detail in Secs. 12.2–12.4.)

Solution. From (1) and (2) we see that $p = 1$, $q = 0$, $r = 1$ in (1), and $a = 0$, $b = \pi$, $k_1 = l_1 = 1$, $k_2 = l_2 = 0$ in (2). For negative $\lambda = -\nu^2$ a general solution of the ODE in (3) is $y(x) = c_1e^{\nu x} + c_2e^{-\nu x}$. From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y \equiv 0$, which is not an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = \nu^2$ a general solution is

$$y(x) = A \cos \nu x + B \sin \nu x.$$

From the first boundary condition we obtain $y(0) = A = 0$. The second boundary condition then yields

$$y(\pi) = B \sin \nu\pi = 0, \qquad \text{thus} \qquad \nu = 0, \pm 1, \pm 2, \cdots.$$

For $\nu = 0$ we have $y \equiv 0$. For $\lambda = \nu^2 = 1, 4, 9, 16, \cdots$, taking $B = 1$, we obtain

$$y(x) = \sin \nu x \qquad (\nu = 1, 2, \cdots).$$

Hence the eigenvalues of the problem are $\lambda = \nu^2$, where $\nu = 1, 2, \cdots$, and corresponding eigenfunctions are $y(x) = \sin \nu x$, where $\nu = 1, 2, \cdots$. ■

Existence of Eigenvalues

Eigenvalues of a Sturm–Liouville problem (1), (2), even infinitely many, exist under rather general conditions on p, q, r in (1). (Sufficient are the conditions in Theorem 1, below, together with $p(x) > 0$ and $r(x) > 0$ on $a < x < b$. Proofs are complicated; see Ref. [A3] or [A11] listed in App. 1.)

Reality of Eigenvalues

Furthermore, if p, q, r, and p' in (1) are real-valued and continuous on the interval $a \leqq x \leqq b$ and r is positive throughout that interval (or negative throughout that interval), then all the eigenvalues of the Sturm–Liouville problem (1), (2) are real. (Proof in App. 4.) This is what the engineer would expect since eigenvalues are often related to frequencies, energies, or other physical quantities that must be real.

Orthogonality

The most remarkable and important property of eigenfunctions of Sturm–Liouville problems is their orthogonality, which will be crucial in series developments in terms of eigenfunctions.

DEFINITION

Orthogonality

Functions $y_1(x), y_2(x), \cdots$ defined on some interval $a \leqq x \leqq b$ are called **orthogonal** on this interval with respect to the **weight function** $r(x) > 0$ if for all m and all n different from m,

$$\int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0 \qquad (m \neq n). \tag{4}$$

The **norm** $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{\int_a^b r(x) y_m^{\,2}(x)\, dx}\,. \tag{5}$$

Note that this is the square root of the integral in (4) with $n = m$.

The functions $y_1, y_2, \cdots$ are called **orthonormal** on $a \leqq x \leqq b$ if they are orthogonal on this interval and all have norm 1.

If $r(x) = 1$, we more briefly call the functions *orthogonal* instead of orthogonal with respect to $r(x) = 1$; similarly for orthonormality. Then

$$\int_a^b y_m(x)\, y_n(x)\, dx = 0 \quad (m \neq n), \qquad \|y_m\| = \sqrt{\int_a^b y_m^{\,2}(x)\, dx}\,.$$

EXAMPLE 3 **Orthogonal Functions. Orthonormal Functions**

The functions $y_m(x) = \sin mx$, $m = 1, 2, \cdots$ form an orthogonal set on the interval $-\pi \leqq x \leqq \pi$, because for $m \neq n$ we obtain by integration [see (11) in App. A3.1]

$$\int_{-\pi}^{\pi} y_m(x) y_n(x)\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x\, dx = 0.$$

The norm $\|y_m\|$ equals $\sqrt{\pi}$, because

$$\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi \qquad (m = 1, 2, \cdots).$$

Hence the corresponding orthonormal set, obtained by division by the norm, is

$$\frac{\sin x}{\sqrt{\pi}}, \qquad \frac{\sin 2x}{\sqrt{\pi}}, \qquad \frac{\sin 3x}{\sqrt{\pi}}, \qquad \cdots.$$

■

Orthogonality of Eigenfunctions

THEOREM 1

Orthogonality of Eigenfunctions

Suppose that the functions p, q, r, and p' in the Sturm–Liouville equation (1) *are real-valued and continuous and $r(x) > 0$ on the interval $a \leqq x \leqq b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm–Liouville problem* (1), (2) *that correspond to different eigenvalues λ_m and λ_n, respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight function r, that is,*

$$\int_a^b r(x)y_m(x)y_n(x)\,dx = 0 \qquad (m \neq n). \tag{6}$$

If $p(a) = 0$, then (2a) *can be dropped from the problem. If $p(b) = 0$, then* (2b) *can be dropped.* [It is then required that y and y' remain bounded at such a point, and the problem is called **singular,** as opposed to a **regular problem** in which (2) is used.]

If $p(a) = p(b)$, then (2) *can be replaced by the* **"periodic boundary conditions"**

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{7}$$

The boundary value problem consisting of the Sturm–Liouville equation (1) and the periodic boundary conditions (7) is called a **periodic Sturm–Liouville problem.**

PROOF By assumption, y_m and y_n satisfy the Sturm–Liouville equations

$$(py_m')' + (q + \lambda_m r)y_m = 0$$
$$(py_n')' + (q + \lambda_n r)y_n = 0$$

respectively. We multiply the first equation by y_n, the second by $-y_m$, and add,

$$(\lambda_m - \lambda_n)ry_my_n = y_m(py_n')' - y_n(py_m')' = [(py_n')y_m - (py_m')y_n]'$$

where the last equality can be readily verified by performing the indicated differentiation of the last expression in brackets. This expression is continuous on $a \leqq x \leqq b$ since p and p' are continuous by assumption and y_m, y_n are solutions of (1). Integrating over x from a to b, we thus obtain

$$(\lambda_m - \lambda_n)\int_a^b ry_my_n\,dx = \Big[p(y_n'y_m - y_m'y_n)\Big]_a^b \qquad (a < b). \tag{8}$$

The expression on the right equals the sum of the subsequent Lines 1 and 2,

$$\begin{aligned} &p(b)\big[y_n'(b)y_m(b) - y_m'(b)y_n(b)\big] && \text{(Line 1)} \\ &-p(a)\big[y_n'(a)y_m(a) - y_m'(a)y_n(a)\big] && \text{(Line 2).} \end{aligned} \tag{9}$$

Hence if (9) is zero, (8) with $\lambda_m - \lambda_n \neq 0$ implies the orthogonality (6). Accordingly, we have to show that (9) is zero, using the boundary conditions (2) as needed.

Case 1. $p(a) = p(b) = 0$**.** Clearly, (9) is zero, and (2) is not needed.

Case 2. $p(a) \neq 0, p(b) = 0$**.** Line 1 of (9) is zero. Consider Line 2. From (2a) we have

$$k_1 y_n(a) + k_2 y_n'(a) = 0,$$
$$k_1 y_m(a) + k_2 y_m'(a) = 0.$$

Let $k_2 \neq 0$. We multiply the first equation by $y_m(a)$, the last by $-y_n(a)$ and add,

$$k_2[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0.$$

This is k_2 times Line 2 of (9), which thus is zero since $k_2 \neq 0$. If $k_2 = 0$, then $k_1 \neq 0$ by assumption, and the argument of proof is similar.

Case 3. $p(a) = 0, p(b) \neq 0$**.** Line 2 of (9) is zero. From (2b) it follows that Line 1 of (9) is zero; this is similar to Case 2.

Case 4. $p(a) \neq 0, p(b) \neq 0$**.** We use both (2a) and (2b) and proceed as in Cases 2 and 3.

Case 5. $p(a) = p(b)$**.** Then (9) becomes

$$p(b)\left[y_n'(b)y_m(b) - y_m'(b)y_n(b) - y_n'(a)y_m(a) + y_m'(a)y_n(a)\right].$$

The expression in brackets $[\cdot\cdot\cdot]$ is zero, either by (2) used as before, or more directly by (7). Hence in this case, (7) can be used instead of (2), as claimed. This completes the proof of Theorem 1. ■

EXAMPLE 4 Application of Theorem 1. Vibrating Elastic String

The ODE in Example 2 is a Sturm–Liouville equation with $p = 1$, $q = 0$, and $r = 1$. From Theorem 1 it follows that the eigenfunctions $y_m = \sin mx$ $(m = 1, 2, \cdots)$ are orthogonal on the interval $0 \leqq x \leqq \pi$. ■

EXAMPLE 5 Application of Theorem 1. Orthogonality of the Legendre Polynomials

Legendre's equation is a Sturm–Liouville equation (see Example 1)

$$\left[(1 - x^2)y'\right]' + \lambda y = 0, \qquad \lambda = n(n + 1)$$

with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a ***singular Sturm—Liouville problem*** on the interval $-1 \leqq x \leqq 1$. We know that for $n = 0, 1, \cdots$, hence $\lambda = 0, 1\cdot 2, 2\cdot 3, \cdots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$\text{(10)} \qquad \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad (m \neq n).$$ ■

EXAMPLE 6 Application of Theorem 1. Orthogonality of the Bessel Functions $J_n(x)$

The Bessel function $J_n(\tilde{x})$ with fixed integer $n \geqq 0$ satisfies Bessel's equation (Sec. 5.5)

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2)J_n(\tilde{x}) = 0,$$

where $\dot{J}_n = dJ_n/d\tilde{x}$, $\ddot{J}_n = d^2J_n/d\tilde{x}^2$. In Example 1 we transformed this equation, by setting $\tilde{x} = kx$, into a Sturm–Liouville equation

$$[xJ_n'(kx)]' + \left(-\frac{n^2}{x} + k^2x\right) J_n(kx) = 0$$

with $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and parameter $\lambda = k^2$. Since $p(0) = 0$, Theorem 1 implies orthogonality on an interval $0 \leqq x \leqq R$ (R given, fixed) of those solutions $J_n(kx)$ that are zero at $x = R$, that is,

$$\text{(11)} \qquad J_n(kR) = 0 \qquad (n \text{ fixed}).$$

[Note that $q(x) = -n^2/x$ is discontinuous at 0, but this does not affect the proof of Theorem 1.] It can be shown (see Ref. [A13]) that $J_n(\tilde{x})$ has infinitely many zeros, say, $\tilde{x} = \alpha_{n,1} < \alpha_{n,2} < \cdots$ (see Fig. 107 in Sec. 5.5 for $n = 0$ and 1). Hence we must have

$$kR = \alpha_{n,m} \qquad \text{thus} \qquad k_{n,m} = \alpha_{n,m}/R \qquad (m = 1, 2, \cdots). \tag{12}$$

This proves the following orthogonality property.

THEOREM 2

Orthogonality of Bessel Functions

For each fixed nonnegative integer n the sequence of Bessel functions of the first kind $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $\cdots$ with $k_{n,m}$ as in (12) forms an orthogonal set on the interval $0 \leqq x \leqq R$ with respect to the weight function $r(x) = x$, that is,

$$\int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x)\,dx = 0 \qquad (j \neq m,\ n \text{ fixed}). \tag{13}$$

Hence we have obtained ***infinitely many orthogonal sets***, each corresponding to one of the ***fixed*** values n. This also illustrates the importance of the zeros of the Bessel functions. ■

EXAMPLE 7 **Eigenvalues from Graphs**

Solve the Sturm–Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(\pi) - y'(\pi) = 0$.

Solution. A general solution and its derivative are

$$y = A\cos kx + B\sin kx \qquad \text{and} \qquad y' = -Ak\sin kx + Bk\cos kx, \qquad k = \sqrt{\lambda}.$$

The first boundary condition gives $y(0) + y'(0) = A + Bk = 0$, hence $A = -Bk$. The second boundary condition and substitution of $A = -Bk$ give

$$\begin{aligned} y(\pi) - y'(\pi) &= A\cos\pi k + B\sin\pi k + Ak\sin\pi k - Bk\cos\pi k \\ &= -Bk\cos\pi k + B\sin\pi k - Bk^2\sin\pi k - Bk\cos\pi k = 0. \end{aligned}$$

We must have $B \neq 0$ since otherwise $B = A = 0$, hence $y = 0$, which is not an eigenfunction. Division by $B\cos\pi k$ gives

$$-k + \tan\pi k - k^2\tan\pi k - k = 0, \qquad \text{thus} \qquad \tan\pi k = \frac{-2k}{k^2 - 1}.$$

The graph in Fig. 110 now shows us where to look for eigenvalues. These correspond to the k-values of the points of intersection of $\tan\pi k$ and the right side $-2k/(k^2 - 1)$ of the last equation. The eigenvalues are $\lambda_m = k_m^{\ 2}$, where $\lambda_0 = 0$ with eigenfunction $y_0 = 1$ and the other λ_m are located near $2^2, 3^2, 4^2, \cdots$, with eigenfunctions $\cos k_m x$ and $\sin k_m x$, $m = 1, 2, \cdots$. The precise numeric determination of the eigenvalues would require a root-finding method (such as those given in Sec. 19.2). ■

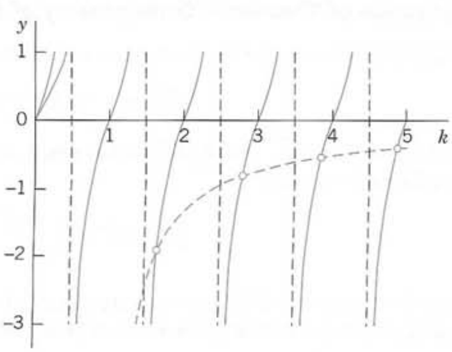


Fig. 110. Example 7. Circles mark the intersections of $\tan\pi k$ and $-2k/(k^2 - 1)$

PROBLEM SET 5.7

1. **(Proof of Theorem 1)** Carry out the details in Cases 3 and 4.
2. **Normalization of eigenfunctions** y_m of (1), (2) means that we multiply y_m by a nonzero constant c_m such that $c_m y_m$ has norm 1. Show that $z_m = cy_m$ with *any* $c \neq 0$ is an eigenfunction for the eigenvalue corresponding to y_m.
3. **(Change of x)** Show that if the functions $y_0(x)$, $y_1(x)$, $\cdots$ form an orthogonal set on an interval $a \leqq x \leqq b$ (with $r(x) = 1$), then the functions $y_0(ct + k)$, $y_1(ct + k)$, $\cdots$, $c > 0$, form an orthogonal set on the interval $(a - k)/c \leqq t \leqq (b - k)/c$.
4. **(Change of x)** Using Prob. 3, derive the orthogonality of 1, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$, $\cdots$ on $-1 \leqq x \leqq 1$ $(r(x) = 1)$ from that of 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$ on $-\pi \leqq x \leqq \pi$.
5. **(Legendre polynomials)** Show that the functions $P_n(\cos \theta)$, $n = 0, 1, \cdots$, form an orthogonal set on the interval $0 \leqq \theta \leqq \pi$ with respect to the weight function $\sin \theta$.
6. **(Tranformation to Sturm–Liouville form)** Show that $y'' + fy' + (g + \lambda h)y = 0$ takes the form (1) if you set $p = \exp(\int f\,dx)$, $q = pg$, $r = hp$. Why would you do such a transformation?

7–19 STURM–LIOUVILLE PROBLEMS

Write the given ODE in the form (1) if it is in a different form. (Use Prob. 6.) Find the eigenvalues and eigenfunctions. Verify orthogonality. (Show the details of your work.)

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(5) = 0$
8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$
9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$
10. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$
11. $y'' + \lambda y = 0, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$
12. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0,$ $y(1) + y'(1) = 0$
13. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$
14. $(xy')' + \lambda x^{-1} y = 0, \quad y(1) = 0, \quad y'(e) = 0.$ (Set $x = e^t$.)
15. $(x^{-1}y')' + (\lambda + 1)x^{-3}y = 0, \quad y(1) = 0,$ $y(e^{\pi}) = 0.$ (Set $x = e^t$.)
16. $y'' - 2y' + (\lambda + 1)y = 0, \quad y(0) = 0,$ $y(1) = 0$
17. $y'' + 8y' + (\lambda + 16)y = 0, \quad y(0) = 0,$ $y(\pi) = 0$
18. $xy'' + 2y' + \lambda xy = 0, \quad y(\pi) = 0, \quad y(2\pi) = 0.$ (Use a CAS or set $y = x^{-1}u$.)
19. $y'' - 2x^{-1}y' + (k^2 + 2x^{-2})y = 0$, $y(1) = 0$, $y(2) = 0$. (Use a CAS or set $y = xu$.)
20. **TEAM PROJECT. Special Functions. Orthogonal polynomials** play a great role in applications. For this reason, Legendre polynomials and various other orthogonal polynomials have been studied extensively; see Refs. [GR1], [GR10] in App. 1. Consider some of the most important ones as follows.

(a) Chebyshev polynomials[10] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$

$$U_n(x) = \frac{\sin[(n + 1)\arccos x]}{\sqrt{1 - x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$T_0 = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x.$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1 - x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[11] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!} \frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$
$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x} x^k L_n\,dx = 0$ for $k < n$. Do this by k integrations by parts.

[10] PAFNUTI CHEBYSHEV (1821–1894), Russian mathematician, is known for his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

[11] EDMOND LAGUERRE (1834–1886), French mathematician, who did research work in geometry and in the theory of infinite series.

5.8 Orthogonal Eigenfunction Expansions

Orthogonal functions (obtained from Sturm–Liouville problems or otherwise) yield important series developments of given functions, as we shall see. This includes the famous *Fourier series* (to which we devote Chaps. 11 and 12), the daily bread of the physicist and engineer for solving problems in heat conduction, mechanical and electrical vibrations, etc. Indeed, orthogonality is one of the most useful ideas ever introduced in applied mathematics.

Standard Notation for Orthogonality and Orthonormality

The integral (4) in Sec. 5.7 defining orthogonality is denoted by (y_m, y_n). This is standard. Also, **Kronecker's delta**[12] δ_{mn} is defined by $\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$ (thus $\delta_{nn} = 1$). Hence for orthonormal functions $y_0, y_1, y_2, \cdots$ with respect to weight $r(x)$ (> 0) on $a \leqq x \leqq b$ we can now simply write $(y_m, y_n) = \delta_{mn}$, written out

$$(1) \qquad (y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n. \end{cases}$$

Also, for the norm we can now write

$$(2) \qquad \|y\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) {y_m}^2(x)\, dx}\,.$$

Write down a few examples of your own, to get used to this practical short notation.

Orthogonal Series

Now comes the instant that shows why orthogonality is a fundamental concept. Let $y_0, y_1, y_2, \cdots$ be an orthogonal set with respect to weight $r(x)$ on an interval $a \leqq x \leqq b$. Let $f(x)$ be a function that can be represented by a convergent series

$$(3) \qquad f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots.$$

This is called an **orthogonal expansion** or **generalized Fourier series.** If the y_m are eigenfunctions of a Sturm–Liouville problem, we call (3) an **eigenfunction expansion.** In (3) we use again m for summation since n will be used as a fixed order of Bessel functions.

Given $f(x)$, we have to determine the coefficients in (3), called the **Fourier constants** *of* $f(x)$ *with respect to* $y_0, y_1, \cdots$. Because of the orthogonality this is simple. All we have to do is to multiply both sides of (3) by $r(x)y_n(x)$ (n ***fixed***) and then integrate on both sides from a to b. We assume that term-by-term integration is permissible. (This is justified, for instance, in the case of "uniform convergence," as is shown in Sec. 15.5.) Then we obtain

$$(f, y_n) = \int_a^b r f y_n\, dx = \int_a^b r \left(\sum_{m=0}^{\infty} a_m y_m \right) y_n\, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n).$$

[12]LEOPOLD KRONECKER (1823–1891). German mathematician at Berlin University, who made important contributions to algebra, group theory, and number theory.

Because of the orthogonality all the integrals on the right are zero, except when $m = n$. Hence the whole infinite series reduces to the single term

$$a_n(y_n, y_n) = a_n \|y_n\|^2.$$

Assuming that all the functions y_n have nonzero norm, we can divide by $\|y_n\|^2$; writing again m for n, to be in agreement with (3), we get the desired formula for the Fourier constants

$$\text{(4)} \qquad a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx \qquad (m = 0, 1, \cdots).$$

EXAMPLE 1 Fourier Series

A most important class of eigenfunction expansions is obtained from the periodic Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(\pi) = y(-\pi), \qquad y'(\pi) = y'(-\pi).$$

A general solution of the ODE is $y = A \cos kx + B \sin kx$, where $k = \sqrt{\lambda}$. Substituting y and its derivative into the boundary conditions, we obtain

$$A \cos k\pi + B \sin k\pi = A \cos(-k\pi) + B \sin(-k\pi)$$

$$-kA \sin k\pi + kB \cos k\pi = -kA \sin(-k\pi) + kB \cos(-k\pi).$$

Since $\cos(-\alpha) = \cos\alpha$, the cosine terms cancel, so that these equations give no condition for these terms. Since $\sin(-\alpha) = -\sin\alpha$, the equations gives the condition $\sin k\pi = 0$, hence $k\pi = m\pi$, $k = m = 0, 1, 2, \cdots$, so that the eigenfunctions are

$$\cos 0 = 1, \qquad \cos x, \qquad \sin x, \qquad \cos 2x, \qquad \sin 2x, \cdots, \qquad \cos mx, \qquad \sin mx, \cdots$$

corresponding pairwise to the eigenvalues $\lambda = k^2 = 0, 1, 4, \cdots, m^2, \cdots$. ($\sin 0 = 0$ is not an eigenfunction.)

By Theorem 1 in Sec. 5.7, any two of these belonging to different eigenvalues are orthogonal on the interval $-\pi \leqq x \leqq \pi$ (note that $r(x) = 1$ for the present ODE). The orthogonality of $\cos mx$ and $\sin mx$ for the same m follows by integration,

$$\int_{-\pi}^{\pi} \cos mx \sin mx\, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx\, dx = 0.$$

For the norms we get $\|1\| = \sqrt{2\pi}$, and $\sqrt{\pi}$ for all the others, as you may verify by integrating 1, $\cos^2 x$, $\sin^2 x$, etc., from $-\pi$ to π. This gives the series (with a slight extension of notation since we have two functions for each eigenvalue 1, 4, 9, $\cdots$)

$$\text{(5)} \qquad f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

According to (4) the coefficients (with $m = 1, 2, \cdots$) are

$$\text{(6)} \qquad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx.$$

The series (5) is called the **Fourier series** of $f(x)$. Its coefficients are called the **Fourier coefficients** of $f(x)$, as given by the so-called **Euler formulas** (6) (not to be confused with the Euler formula (11) in Sec. 2.2).

For instance, for the **"periodic rectangular wave"** in Fig. 111, given by

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x),$$

we get from (6) the values $a_0 = 0$ and

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos mx\,dx + \int_{0}^{\pi} 1\cdot\cos mx\,dx\right] = 0,$$

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin mx\,dx + \int_{0}^{\pi} 1\cdot\sin mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos mx}{m}\bigg|_{-\pi}^{0} - \frac{\cos mx}{m}\bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi m}[1 - 2\cos m\pi + 1] = \begin{cases} 4/(\pi m) & \text{if } m = 1, 3, \cdots, \\ 0 & \text{if } m = 2, 4 \cdots. \end{cases}$$

Hence the Fourier series of the periodic rectangular wave is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

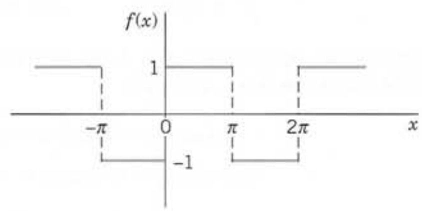


Fig. 111. Periodic rectangular wave in Example 1

Fourier series are by far the most important eigenfunction expansions, so important to the engineer that we shall devote two chapters (11 and 12) to them and their applications, and discuss numerous examples.

Did it surprise you that a series of continuous functions (sine functions) can represent a discontinuous function? More on this in Chap. 11.

EXAMPLE 2 Fourier–Legendre Series

A **Fourier–Legendre series** is an eigenfunction expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2(\tfrac{3}{2}x^2 - \tfrac{1}{2}) + \cdots$$

in terms of Legendre polynomials (Sec. 5.3). The latter are the eigenfunctions of the Sturm–Liouville problem in Example 5 of Sec. 5.7 on the interval $-1 \leqq x \leqq 1$. We have $r(x) = 1$ for Legendre's equation, and (4) gives

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx, \qquad m = 0, 1, \cdots \tag{7}$$

because the norm is

$$\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2\,dx} = \sqrt{\frac{2}{2m+1}} \qquad (m = 0, 1, \cdots) \tag{8}$$

as we state without proof. (The proof is tricky; it uses Rodrigues's formula in Problem Set 5.3 and a reduction of the resulting integral to a quotient of gamma functions.)

For instance, let $f(x) = \sin \pi x$. Then we obtain the coefficients

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} (\sin \pi x) P_m(x)\, dx, \qquad \text{thus} \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x\, dx = \frac{3}{\pi} = 0.95493, \qquad \text{etc.}$$

Hence the Fourier–Legendre series of $\sin \pi x$ is

$$\sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + 0.21429 P_5(x) - 0.01664 P_7(x) + 0.00068 P_9(x) - 0.00002 P_{11}(x) + \cdots.$$

The coefficient of P_{13} is about $3 \cdot 10^{-7}$. The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve. Can you see why the even-numbered coefficients are zero? Why a_3 is the absolutely biggest coefficient? ■

EXAMPLE 3 Fourier–Bessel Series

In Example 6 of Sec. 5.7 we obtained infinitely many orthogonal sets of Bessel functions, one for each of J_0, $J_1, J_2, \cdots$. Each set is orthogonal on an interval $0 \leqq x \leqq R$ with a fixed positive R of our choice and with respect to the weight x. The orthogonal set for J_n is $J_n(k_{n,1}x), J_n(k_{n,2}x), J_n(k_{n,3}x), \cdots$, where n is ***fixed*** and $k_{n,m}$ is given in (12), Sec. 5.7. The corresponding Fourier–Bessel series is

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \qquad (n \text{ fixed}). \tag{9}$$

The coefficients are (with $\alpha_{n,m} = k_{n,m}R$)

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x f(x)\, J_n(k_{n,m}x)\, dx, \qquad m = 1, 2, \cdots \tag{10}$$

because the square of the norm is

$$\| J_n(k_{n,m}x) \|^2 = \int_0^R x J_n{}^2(k_{n,m}x)\, dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R) \tag{11}$$

as we state without proof (which is tricky; see the discussion beginning on p. 576 of [A13]).

For instance, let us consider $f(x) = 1 - x^2$ and take $R = 1$ and $n = 0$ in the series (9), simply writing λ for $\alpha_{0,m}$. Then $k_{n,m} = \alpha_{0,m} = \lambda = 2.405, 5.520, 8.654, 11.792$, etc. (use a CAS or Table A1 in App. 5). Next we calculate the coefficients a_m by (10),

$$a_m = \frac{2}{J_1{}^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx.$$

This can be integrated by a CAS or by formulas as follows. First use $[xJ_1(\lambda x)]' = \lambda x J_0(\lambda x)$ from Theorem 3 in Sec. 5.5 and then integration by parts,

$$a_m = \frac{2}{J_1{}^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx = \frac{2}{J_1{}^2(\lambda)} \left[\frac{1}{\lambda}(1 - x^2) x J_1(\lambda x) \Big|_0^1 - \frac{1}{\lambda} \int_0^1 x J_1(\lambda x)(-2x)\, dx \right].$$

The integral-free part is zero. The remaining integral can be evaluated by $\left[x^2 J_2(\lambda x)\right]' = \lambda x^2 J_1(\lambda x)$ from Theorem 3 in Sec. 5.5. This gives

$$a_m = \frac{4 J_2(\lambda)}{\lambda^2 J_1{}^2(\lambda)} \qquad (\lambda = \alpha_{0,m}).$$

Numeric values can be obtained from a CAS (or from the table on p. 409 of Ref. [GR1] in App. 1, together with the formula $J_2 = 2x^{-1}J_1 - J_0$ in Theorem 3 of Sec. 5.5). This gives the eigenfunction expansion of $1 - x^2$ in terms of Bessel functions J_0, that is,

$$1 - x^2 = 1.1081 J_0(2.405x) - 0.1398 J_0(5.520x) + 0.0455 J_0(8.654x) - 0.0210 J_0(11.792x) + \cdots.$$

A graph would show that the curve of $1 - x^2$ and that of the sum of the first three terms practically coincide. ■

Mean Square Convergence. Completeness of Orthonormal Sets

The remaining part of this section will give an introduction to a convergence suitable in connection with orthogonal series and quite different from the convergence used in calculus for Taylor series.

In practice, one uses only orthonormal sets that consist of "sufficiently many" functions, so that one can represent large classes of functions by a generalized Fourier series (3)—certainly all continuous functions on an interval $a \leqq x \leqq b$, but also functions that do "not have too many" discontinuities (see Example 1). Such orthonormal sets are called "complete" (in the set of functions considered; definition below). For instance, the orthonormal sets corresponding to Examples 1–3 are complete in the set of functions continuous on the intervals considered (or even in more general sets of functions; see Ref. [GR7], Secs. 3.4–3.7, listed in App. 1, where "complete sets" bear the more modern name "total sets").

In this connection, convergence is **convergence in the norm,** also called **mean-square convergence;** that is, a sequence of functions f_k is called **convergent** *with the limit* f if

$$\lim_{k\to\infty} \|f_k - f\| = 0; \tag{12*}$$

written out by (2) (where we can drop the square root, as this does not affect the limit)

$$\lim_{k\to\infty} \int_a^b r(x)[f_k(x) - f(x)]^2 \, dx = 0. \tag{12}$$

Accordingly, the series (3) converges and represents f if

$$\lim_{k\to\infty} \int_a^b r(x)[s_k(x) - f(x)]^2 \, dx = 0 \tag{13}$$

where s_k is the kth partial sum of (3),

$$s_k(x) = \sum_{m=0}^{k} a_m y_m(x). \tag{14}$$

By definition, an orthonormal set $y_0, y_1, \cdots$ on an interval $a \leqq x \leqq b$ is **complete** *in a set of functions S* defined on $a \leqq x \leqq b$ if we can approximate every f belonging to S arbitrarily closely by a linear combination $a_0 y_0 + a_1 y_1 + \cdots + a_k y_k$, that is, technically, if for every $\epsilon > 0$ we can find constants $a_0, \cdots, a_k$ (with k large enough) such that

$$\|f - (a_0 y_0 + \cdots + a_k y_k)\| < \epsilon. \tag{15}$$

An interesting and basic consequence of the integral in (13) is obtained as follows. Performing the square and using (14), we first have

$$\begin{aligned}\int_a^b r(x)[s_k(x) - f(x)]^2 \, dx &= \int_a^b r s_k^{\,2} \, dx - 2\int_a^b r f s_k \, dx + \int_a^b r f^2 \, dx \\ &= \int_a^b r \left[\sum_{m=0}^{k} a_m y_m\right]^2 dx - 2\sum_{m=0}^{k} a_m \int_a^b r f y_m \, dx + \int_a^b r f^2 \, dx.\end{aligned}$$

The first integral on the right equals $\Sigma\, a_m^{\,2}$ because $\int r y_m y_l \, dx = 0$ for $m \neq l$, and $\int r y_m^{\,2} \, dx = 1$. In the second sum on the right, the integral equals a_m, by (4) with

$\|y_m\|^2 = 1$. Hence the first term on the right cancels half of the second term, so that the right side reduces to

$$-\sum_{m=0}^{k} a_m^{\,2} + \int_a^b rf^2\,dx.$$

This is nonnegative because in the previous formula the integrand on the left is nonnegative (recall that the weight $r(x)$ is positive!) and so is the integral on the left. This proves the important **Bessel's inequality**

$$\sum_{m=0}^{k} a_m^{\,2} \leqq \|f\|^2 = \int_a^b r(x)f(x)^2\,dx \qquad (k = 1, 2, \cdots). \tag{16}$$

Here we can let $k \to \infty$, because the left sides form a monotone increasing sequence that is bounded by the right side, so that we have convergence by the familiar Theorem 1 in App. A3.3. Hence

$$\sum_{m=0}^{\infty} a_m^{\,2} \leqq \|f\|^2. \tag{17}$$

Furthermore, if $y_0, y_1, \cdots$ is complete in a set of functions S, then (13) holds for every f belonging to S. By (15) this implies equality in (16) with $k \to \infty$. Hence in the case of completeness every f in S satisfies the so-called **Parseval's equality**

$$\sum_{m=0}^{\infty} a_m^{\,2} = \|f\|^2 = \int_a^b r(x)f(x)^2\,dx. \tag{18}$$

As a consequence of (18) we prove that in the case of *completeness* there is no function orthogonal to *every* function of the orthonormal set, with the trivial exception of a function of zero norm:

THEOREM 1 **Completeness**

Let $y_0, y_1, \cdots$ be a complete orthonormal set on $a \leqq x \leqq b$ in a set of functions S. Then if a function f belongs to S and is orthogonal to every y_m, it must have norm zero. In particular, if f is continuous, then f must be identically zero.

PROOF Since f is orthogonal to every y_m, the left side of (18) must be zero. If f is continuous, then $\|f\| = 0$ implies $f(x) \equiv 0$, as can be seen directly from (2) with f instead of y_m because $r(x) > 0$. ■

EXAMPLE 4 **Fourier Series**

The orthonormal set in Example 1 is complete in the set of continuous functions on $-\pi \leqq x \leqq \pi$. Verify directly that $f(x) \equiv 0$ is the only continuous function orthogonal to all the functions of that set.

Solution. Let f be any continuous function. By the orthogonality (we can omit $\sqrt{2\pi}$ and $\sqrt{\pi}$),

$$\int_{-\pi}^{\pi} 1 \cdot f(x)\,dx = 0, \qquad \int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0, \qquad \int_{-\pi}^{\pi} f(x)\sin mx\,dx = 0.$$

Hence $a_m = 0$ and $b_m = 0$ in (6) for all m, so that (3) reduces to $f(x) \equiv 0$. ■

This is the end of Chap. 5 on the power series method and the Frobenius method, which are indispensable in solving linear ODEs with variable coefficients, some of the most important of which we have discussed and solved. We have also seen that the latter are important sources of special functions having orthogonality properties that make them suitable for orthogonal series representations of given functions.

PROBLEM SET 5.8

1–4 FOURIER–LEGENDRE SERIES

Showing the details of your calculations, develop:

1. $7x^4 - 6x^2$
2. $(x + 1)^2$
3. $x^3 - x^2 + x - 1$
4. $1, x, x^2, x^3$

5. Prove that if $f(x)$ in Example 2 is even [that is, $f(x) = f(-x)$], its series contains only $P_m(x)$ with even m.

6–16 CAS EXPERIMENTS. FOURIER–LEGENDRE SERIES

Find and graph (on common axes) the partial sums up to that S_{m_0} whose graph practically coincides with that of $f(x)$ within graphical accuracy. State what m_0 is. On what does the size of m_0 seem to depend?

6. $f(x) = \sin \pi x$
7. $f(x) = \sin 2\pi x$
8. $f(x) = \cos \pi x$
9. $f(x) = \cos 2\pi x$
10. $f(x) = \cos 3\pi x$
11. $f(x) = e^x$
12. $f(x) = e^{-x^2}$
13. $f(x) = 1/(1 + x^2)$

14. $f(x) = J_0(\alpha_{0,1}x)$, where $\alpha_{0,1}$ is the first positive zero of J_0

15. $f(x) = J_0(\alpha_{0,2}x)$, where $\alpha_{0,2}$ is the second positive zero of J_0

16. $f(x) = J_1(\alpha_{1,1}x)$, where $\alpha_{1,1}$ is the first positive zero of J_1

17. CAS EXPERIMENT. Fourier–Bessel Series. Use Example 3 and again take $n = 10$ and $R = 1$, so that you get the series

$$(19) \quad f(x) = a_1J_0(\alpha_{0,1}x) + a_2J_0(\alpha_{0,2}x) + a_3J_0(\alpha_{0,3}x) + \cdots$$

with the zeros $\alpha_{0,1}$ $\alpha_{0,2}, \cdots$ from your CAS (see also Table A1 in App. 5).

(a) Graph the terms $J_0(\alpha_{0,1}x), \cdots, J_0(\alpha_{0,10}x)$ for $0 \leqq x \leqq 1$ on common axes.

(b) Write a program for calculating partial sums of (19). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

(c) Take $f(x) = 1$ in (19) and evaluate the integrals for the coefficients analytically by (24a), Sec. 5.5, with $\nu = 1$. Graph the first few partial sums on common axes.

18. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.[13] These orthogonal polynomials are defined by $He_0(1) = 1$ and

$$He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \quad n = 1, 2, \cdots.$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferred in applications.

(a) Small Values of *n*. Show that

$$He_1(x) = x, \qquad He_2(x) = x^2 - 1,$$
$$He_3(x) = x^3 - 3x, \qquad He_4(x) = x^4 - 6x^2 + 3.$$

(b) Generating Function. A generating function of the Hermite polynomials is

$$(20) \qquad e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x)t^n$$

because $He_n(x) = n!\,a_n(x)$. Prove this. *Hint:* Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

(c) Derivative. Differentiating the generating function with respect to x, show that

$$(21) \qquad He_n'(x) = nHe_{n-1}(x).$$

[13] CHARLES HERMITE (1822–1901), French mathematician, is known for his work in algebra and number theory. The great HENRI POINCARÉ (1854–1912) was one of his students.

(d) Orthogonality on the x-Axis needs a weight function that goes to zero sufficiently fast as $x \to \pm\infty$. (Why?) Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint.* Use integration by parts and (21).

(e) ODEs. Show that

$$He_n'(x) = xHe_n(x) - He_{n+1}(x). \quad (22)$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

$$y'' - xy' + ny = 0. \quad (23)$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**[14]

$$w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots). \quad (24)$$

19. WRITING PROJECT. Orthogonality. Write a short report (2–3 pages) about the most important ideas and facts related to orthogonality and orthogonal series and their applications.

CHAPTER 5 REVIEW QUESTIONS AND PROBLEMS

1. What is a power series? Can it contain negative or fractional powers? How would you test for convergence?

2. Why could we use the power series method for Legendre's equation but needed the Frobenius method for Bessel's equation?

3. Why did we introduce two kinds of Bessel functions, J and Y?

4. What is the hypergeometric equation and why did Gauss introduce it?

5. List the three cases of the Frobenius method, giving examples of your own.

6. What is the difference between an initial value problem and a boundary value problem?

7. What does orthogonality of functions mean and how is it used in series expansions? Give examples.

8. What is the Sturm–Liouville theory and its practical importance?

9. What do you remember about the orthogonality of the Legendre polynomials? Of Bessel functions?

10. What is completeness of orthogonal sets? Why is it important?

11–20 SERIES SOLUTIONS

Find a basis of solutions. Try to identify the series as expansions of known functions. (Show the details of your work.)

11. $y'' - 9y = 0$

12. $(1 - x)^2y'' + (1 - x)y' - 3y = 0$

13. $xy'' - (x + 1)y' + y = 0$

14. $x^2y'' - 3xy' + 4y = 0$

15. $y'' + 4xy' + (4x^2 + 2)y = 0$

16. $x^2y'' - 4xy' + (x^2 + 6)y = 0$

17. $xy'' + (2x + 1)y' + (x + 1)y = 0$

18. $(x^2 - 1)y'' - 2xy' + 2y = 0$

19. $(x^2 - 1)y'' + 4xy' + 2y = 0$

20. $x^2y'' + xy' + (4x^4 - 1)y = 0$

21–25 BESSEL'S EQUATION

Find a general solution in terms of Bessel functions. (Use the indicated transformations and show the details.)

21. $x^2y'' + xy' + (36x^2 - 2)y = 0 \qquad (6x = z)$

22. $x^2y'' + 5xy' + (x^2 - 12)y = 0 \qquad (y = u/x^2)$

23. $x^2y'' + xy' + 4(x^4 - 1)y = 0 \qquad (x^2 = z)$

24. $4x^2y'' - 20xy' + (4x^2 + 35)y = 0 \qquad (y = x^3u)$

25. $y'' + k^2x^2y = 0 \qquad (y = u\sqrt{x},\ \tfrac{1}{2}kx^2 = z)$

26–30 BOUNDARY VALUE PROBLEMS

Find the eigenvalues and eigenfunctions.

26. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

27. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$

28. $(xy')' + \lambda x^{-1}y = 0, \quad y(1) = 0, \quad y(e) = 0.$ (Set $x = e^t$.)

29. $x^2y'' + xy' + (\lambda x^2 - 1)y = 0, \quad y(0) = 0, \quad y(1) = 0$

30. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(2\pi) = 0$

31–35 CAS PROBLEMS

Write a program, develop in a Fourier–Legendre series, and graph the first five partial sums on common axes, together with the given function. Comment on accuracy.

31. $e^{2x} \quad (-1 \leqq x \leqq 1)$

32. $\sin(\pi x^2) \quad (-1 \leqq x \leqq 1)$

33. $1/(1 + |x|) \quad (-1 \leqq x \leqq 1)$

34. $|\cos \pi x| \quad (-1 \leqq x \leqq 1)$

35. x if $0 \leqq x \leqq 1$, 0 if $-1 \leqq x < 0$

[14] HEINRICH WEBER (1842–1913), German mathematician.

SUMMARY OF CHAPTER 5
Series Solution of ODEs. Special Functions

The **power series method** gives solutions of linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \quad (1)$$

with ***variable coefficients*** p and q in the form of a power series (with any center x_0, e.g., $x_0 = 0$)

$$y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \quad (2)$$

Such a solution is obtained by substituting (2) and its derivatives into (1). This gives a **recurrence formula** for the coefficients. You may program this formula (or even obtain and graph the whole solution) on your CAS.

If p and q are **analytic** at x_0 (that is, representable by a power series in powers of $x - x_0$ with positive radius of convergence; Sec. 5.2), then (1) has solutions of this form (2). The same holds if $\tilde{h}, \tilde{p}, \tilde{q}$ in

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

are analytic at x_0 and $\tilde{h}(x_0) \neq 0$, so that we can divide by $\tilde{h}$ and obtain the standard form (1). **Legendre's equation** is solved by the power series method in Sec. 5.3.

The **Frobenius method** (Sec. 5.4) extends the power series method to ODEs

$$y'' + \frac{a(x)}{x - x_0} y' + \frac{b(x)}{(x - x_0)^2} y = 0 \quad (3)$$

whose coefficients are **singular** (i.e., not analytic) at x_0, but are "not too bad," namely, such that a and b are analytic at x_0. Then (3) has at least one solution of the form

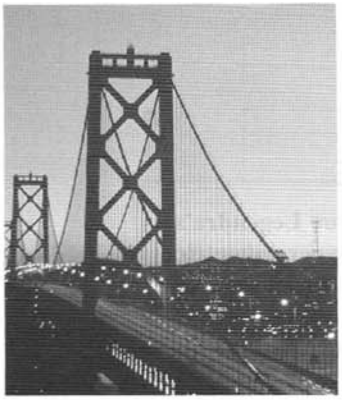
$$y(x) = (x - x_0)^r \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0(x - x_0)^r + a_1(x - x_0)^{r+1} + \cdots \quad (4)$$

where r can be any real (or even complex) number and is determined by substituting (4) into (3) from the **indicial equation** (Sec. 5.4), along with the coefficients of (4). A second linearly independent solution of (3) may be of a similar form (with different r and a_m's) or may involve a logarithmic term. **Bessel's equation** is solved by the Frobenius method in Secs. 5.5 and 5.6.

"Special functions" is a common name for higher functions, as opposed to the usual functions of calculus. Most of them arise either as nonelementary integrals [see (24)–(44) in App. 3.1] or as solutions of (1) or (3). They get a name and notation and are included in the usual CASs if they are important in application or in theory.

Of this kind, and particularly useful to the engineer and physicist, are **Legendre's equation and polynomials P_0, P_1, $\cdots$** (Sec. 5.3), **Gauss's hypergeometric equation and functions $F(a, b, c; x)$** (Sec. 5.4), and **Bessel's equation and functions J_ν and Y_ν** (Secs. 5.5, 5.6).

Modeling involving ODEs usually leads to initial value problems (Chaps. 1–3) or **boundary value problems.** Many of the latter can be written in the form of **Sturm–Liouville problems** (Sec. 5.7). These are **eigenvalue problems** involving a parameter λ that is often related to frequencies, energies, or other physical quantities. Solutions of Sturm–Liouville problems, called **eigenfunctions,** have many general properties in common, notably the highly important **orthogonality** (Sec. 5.7), which is useful in **eigenfunction expansions** (Sec. 5.8) in terms of cosine and sine (*"Fourier series"*, the topic of Chap. 11), Legendre polynomials, Bessel functions (Sec. 5.8), and other eigenfunctions.



CHAPTER 6

Laplace Transforms

The Laplace transform method is a powerful method for solving linear ODEs and corresponding initial value problems, as well as systems of ODEs arising in engineering. The process of solution consists of three steps (see Fig. 112).

Step 1. The given ODE is transformed into an algebraic equation (**"subsidiary equation")**.

Step 2. The subsidiary equation is solved by purely algebraic manipulations.

Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.

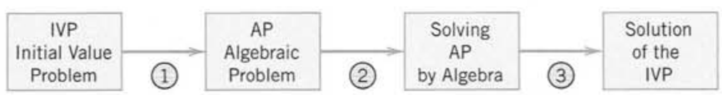


Fig. 112. Solving an IVP by Laplace transforms

Thus solving an ODE is reduced to an *algebraic* problem (plus those transformations). This switching from calculus to algebra is called **operational calculus.** The Laplace transform method is the most important operational method to the engineer. This method has two main advantages over the usual methods of Chaps. 1–4:

A. Problems are solved more directly, initial value problems without first determining a general solution, and nonhomogeneous ODEs without first solving the corresponding homogeneous ODE.

B. More importantly, the use of the **unit step function (Heaviside function** in Sec. 6.3) and **Dirac's delta** (in Sec. 6.4) make the method particularly powerful for problems with inputs (driving forces) that have discontinuities or represent short impulses or complicated periodic functions.

In this chapter we consider the Laplace transform and its application to engineering problems involving ODEs. PDEs will be solved by the Laplace transform in Sec. 12.11.

General formulas are listed in Sec. 6.8, **transforms and inverses** in Sec. 6.9. The usual **CASs** can handle most Laplace transforms.

Prerequisite: Chap. 2
Sections that may be omitted in a shorter course: 6.5, 6.7
References and Answers to Problems: App. 1 Part A, App. 2.

6.1 Laplace Transform. Inverse Transform. Linearity. *s*-Shifting

If $f(t)$ is a function defined for all $t \geqq 0$, its **Laplace transform**[1] is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value). This assumption is usually satisfied in applications—we shall discuss this near the end of the section.

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform.** It is an **"integral transform"**

$$F(s) = \int_0^\infty k(s, t) f(t)\, dt$$

with **"kernel"** $k(s, t) = e^{-st}$.

Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F). \tag{1*}$$

Note that (1) and (1*) together imply $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$.

Notation

Original functions depend on t and their transforms on s—keep this in mind! Original functions are denoted by ***lowercase letters*** and their transforms by the same ***letters in capital,*** so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

EXAMPLE 1 Laplace Transform

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

---

[1] PIERRE SIMON MARQUIS DE LAPLACE (1749–1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions, and probability theory. Napoléon Bonaparte was his student for a year. For Laplace's interesting political involvements, see Ref. [GR2], listed in App. 1.

The powerful practical Laplace transform techniques were developed over a century later by the English electrical engineer OLIVER HEAVISIDE (1850–1925) and were often called "Heaviside calculus."

We shall drop variables when this simplifies formulas without causing confusion. For instance, in (1) we wrote $\mathcal{L}(f)$ instead of $\mathcal{L}(f)(s)$ and in (1*) $\mathcal{L}^{-1}(F)$ instead of $\mathcal{L}^{-1}(F)(t)$.

Our notation is convenient, but we should say a word about it. The interval of integration in (1) is infinite. Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt.$$

Hence our convenient notation means

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s}\, e^{-st} \right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s}\, e^{-sT} + \frac{1}{s}\, e^0 \right] = \frac{1}{s} \qquad (s > 0).$$

We shall use this notation throughout this chapter. ■

EXAMPLE 2 Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{a-s}\, e^{-(s-a)t} \Big|_0^\infty ;$$

hence, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}\,. \qquad ■$$

Must we go on in this fashion and obtain the transform of one function after another directly from the definition? The answer is no. And the reason is that new transforms can be found from known ones by the use of the many general properties of the Laplace transform. Above all, the Laplace transform is a "linear operation," just as differentiation and integration. By this we mean the following.

THEOREM 1 Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

PROOF By the definition in (1),

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}. \qquad ■$$

EXAMPLE 3 Application of Theorem 1: Hyperbolic Functions

Find the transforms of cosh at and sinh at.

Solution. Since $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we obtain from Example 2 and Theorem 1

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}\,. \qquad ■$$

EXAMPLE 4 Cosine and Sine

Derive the formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Calculus. We write $L_c = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain

$$L_c = \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} L_c.$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Transforms Using Derivatives. See next section.

Solution by Complex Methods. In Example 2, if we set $a = i\omega$ with $i = \sqrt{-1}$, we obtain

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{(s - i\omega)(s + i\omega)} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}.$$

Now by Theorem 1 and $e^{i\omega t} = \cos \omega t + i \sin \omega t$ [see (11) in Sec. 2.2 with ωt instead of t] we have

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos \omega t + i \sin \omega t) = \mathcal{L}(\cos \omega t) + i\mathcal{L}(\sin \omega t).$$

If we equate the real and imaginary parts of this and the previous equation, the result follows. (This formal calculation can be justified in the theory of complex integration.) ■

Basic transforms are listed in Table 6.1. We shall see that from these almost all the others can be obtained by the use of the general properties of the Laplace transform. Formulas 1–3 are special cases of formula 4, which is proved by induction. Indeed, it is true for $n = 0$ because of Example 1 and $0! = 1$. We make the induction hypothesis that it holds for any integer $n \geqq 0$ and then get it for $n + 1$ directly from (1). Indeed, integration by parts first gives

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} \, dt = -\frac{1}{s} e^{-st} t^{n+1} \Big|_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n \, dt.$$

Now the integral-free part is zero and the last part is $(n + 1)/s$ times $\mathcal{L}(t^n)$. From this and the induction hypothesis,

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s} \mathcal{L}(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

This proves formula 4.

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

$\Gamma(a + 1)$ in formula 5 is the so-called *gamma function* [(15) in Sec. 5.5 or (24) in App. A3.1]. We get formula 5 from (1), setting $st = x$:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx$$

where $s > 0$. The last integral is precisely that defining $\Gamma(a + 1)$, so we have $\Gamma(a + 1)/s^{a+1}$, as claimed. (CAUTION! $\Gamma(a + 1)$ has x^a in the integral, not x^{a+1}.)

Note the formula 4 also follows from 5 because $\Gamma(n + 1) = n!$ for integer $n \geqq 0$.

Formulas 6–10 were proved in Examples 2–4. Formulas 11 and 12 will follow from 7 and 8 by "shifting," to which we turn next.

s-Shifting: Replacing s by $s - a$ in the Transform

The Laplace transform has the very useful property that if we know the transform of $f(t)$, we can immediately get that of $e^{at} f(t)$, as follows.

THEOREM 2

First Shifting Theorem, s-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s - a)\}.$$

PROOF We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}[e^{at} f(t)]\, dt = \mathscr{L}\{e^{at} f(t)\}.$$

If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem. (CAUTION! $-a$ in $F(s-a)$ but $+a$ in $e^{at} f(t)$.) ∎

EXAMPLE 5 ***s*-Shifting: Damped Vibrations. Completing the Square**

From Example 4 and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 6.1,

$$\mathscr{L}\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2 + \omega^2}, \qquad \mathscr{L}\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}.$$

For instance, use these formulas to find the inverse of the transform

$$\mathscr{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}.$$

Solution. Applying the inverse transform, using its linearity (Prob. 28), and completing the square, we obtain

$$f = \mathscr{L}^{-1}\left\{\frac{3(s+1) - 140}{(s+1)^2 + 400}\right\} = 3\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 20^2}\right\} - 7\mathscr{L}^{-1}\left\{\frac{20}{(s+1)^2 + 20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration (Fig. 113)

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$ ∎

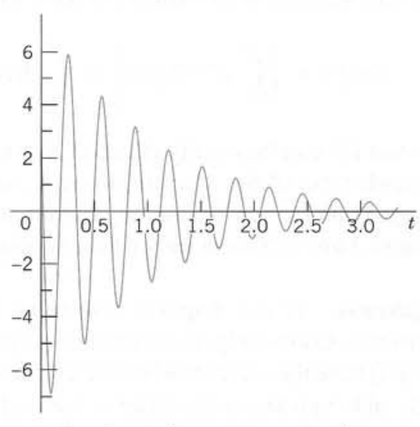


Fig. 113. Vibrations in Example 5

Existence and Uniqueness of Laplace Transforms

This is not a big *practical* problem because in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts.

A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geqq 0$ and some constants M and k it satisfies the **"growth restriction"**

(2) $$|f(t)| \leqq Me^{kt}.$$

(The growth restriction (2) is sometimes called "growth of exponential order," which may be misleading since it hides that the exponent must be kt, not kt^2 or similar.)

$f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. $f(t)$ is **piecewise continuous** on a finite interval $a \leqq t \leqq b$ where f is defined, if this interval can be divided into *finitely many* subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives **finite jumps** as in Fig. 114 as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

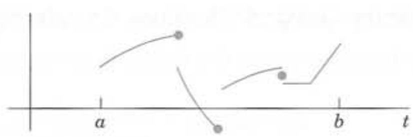


Fig. 114. Example of a piecewise continuous function $f(t)$. (The dots mark the function values at the jumps.)

THEOREM 3

Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.

PROOF Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leqq \int_0^\infty |f(t)| e^{-st}\, dt \leqq \int_0^\infty M e^{kt} e^{-st}\, dt = \frac{M}{s-k}. \qquad \blacksquare$$

Note that (2) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n!e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (2) for any M and k is e^{t^2} (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary (see Prob. 22).

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

PROBLEM SET 6.1

1–20 LAPLACE TRANSFORMS

Find the Laplace transforms of the following functions. Show the details of your work. (a, b, k, ω, θ are constants.)

1. $t^2 - 2t$

2. $(t^2 - 3)^2$

3. $\cos 2\pi t$

4. $\sin^2 4t$

5. $e^{2t} \cosh t$

6. $e^{-t} \sinh 5t$

7. $\cos(\omega t + \theta)$

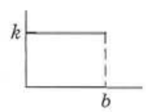
8. $\sin(3t - \frac{1}{2})$

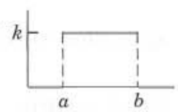
9. e^{3a-2bt}

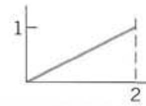
10. $-8 \sin 0.2t$

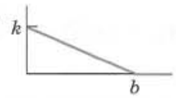
11. $\sin t \cos t$

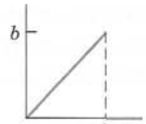
12. $(t + 1)^3$

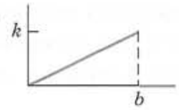
13.


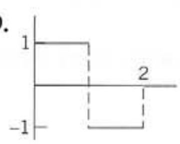
14.


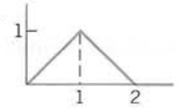
15.


16.


17.


18.


19.


20.


21. Using $\mathcal{L}(f)$ in Prob. 13, find $\mathcal{L}(f_1)$, where $f_1(t) = 0$ if $t \leqq 2$ and $f_1(t) = 1$ if $t > 2$.

22. (Existence) Show that $\mathcal{L}(1/\sqrt{t}) = \sqrt{\pi/s}$. [Use (30) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in App. 3.1.] Conclude from this that the conditions in Theorem 3 are sufficient but not necessary for the existence of a Laplace transform.

23. (Change of scale) If $\mathcal{L}(f(t)) = F(s)$ and c is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$. (*Hint:* Use (1).) Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.

24. (Nonexistence) Show that e^{t^2} does not satisfy a condition of the form (2).

25. (Nonexistence) Give simple examples of functions (defined for all $x \geqq 0$) that have no Laplace transform.

26. (Table 6.1) Derive formula 6 from formulas 9 and 10.

27. (Table 6.1) Convert Table 6.1 from a table for finding transforms to a table for finding inverse transforms (with obvious changes, e.g., $\mathcal{L}^{-1}(1/s^n) = t^{n-1}/(n-1)!$, etc.).

28. (Inverse transform) Prove that $\mathcal{L}^{-1}$ is linear. *Hint.* Use the fact that $\mathcal{L}$ is linear.

29–40 INVERSE LAPLACE TRANSFORMS

Given $F(s) = \mathcal{L}(f)$, find $f(t)$. Show the details. (L, n, k, a, b are constants.)

29. $\dfrac{4s - 3\pi}{s^2 + \pi^2}$

30. $\dfrac{2s + 16}{s^2 - 16}$

31. $\dfrac{s^4 - 3s^2 + 12}{s^5}$

32. $\dfrac{10}{2s + \sqrt{2}}$

33. $\dfrac{n\pi L}{L^2 s^2 + n^2\pi^2}$

34. $\dfrac{20}{(s - 1)(s + 4)}$

35. $\dfrac{8}{s^2 + 4s}$

36. $\displaystyle\sum_{k=1}^{4} \frac{(k + 1)^2}{s + k^2}$

37. $\dfrac{1}{(s - \sqrt{3})(s + \sqrt{5})}$

38. $\dfrac{18s - 12}{9s^2 - 1}$

39. $\dfrac{1}{s^2 + 5} - \dfrac{1}{s + 5}$

40. $\dfrac{1}{(s + a)(s + b)}$

41–54 APPLICATIONS OF THE FIRST SHIFTING THEOREM (s-SHIFTING)

In Probs. 41–46 find the transform. In Probs. 47–54 find the inverse transform. Show the details.

41. $3.8te^{2.4t}$

42. $-3t^4e^{-0.5t}$

43. $5e^{-at} \sin \omega t$

44. $e^{-3t} \cos \pi t$

45. $e^{-kt}(a \cos t + b \sin t)$

46. $e^{-t}(a_0 + a_1 t + \cdots + a_n t^n)$

47. $\dfrac{7}{(s - 1)^3}$

48. $\dfrac{\pi}{(s + \pi)^2}$

49. $\dfrac{\sqrt{8}}{(s + \sqrt{2})^3}$

50. $\dfrac{s - 6}{(s - 1)^2 + 4}$

51. $\dfrac{15}{s^2 + 4s + 29}$

52. $\dfrac{4s - 2}{s^2 - 6s + 18}$

53. $\dfrac{\pi}{s^2 + 10\pi s + 24\pi^2}$

54. $\dfrac{2s - 56}{s^2 - 4s - 12}$

6.2 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms.*** Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $\mathcal{L}(f)$ by s (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives.

THEOREM 1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{1}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Formula (1) holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction (2) in Sec. 6.1 and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly, (2) holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.

PROOF We prove (1) first under the *additional assumption* that f' is continuous. Then by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = [e^{-st} f(t)] \Big|_0^\infty + s \int_0^\infty e^{-st} f(t)\, dt.$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $\mathcal{L}(f')$ exists when $s > k$ and (1) holds.

If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (2) now follows by applying (1) to f'' and then substituting (1), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0). \quad \blacksquare$$

Continuing by substitution as in the proof of (2) and using induction, we obtain the following extension of Theorem 1.

THEOREM 2

Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction (2) in Sec. 6.1. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

EXAMPLE 1 **Transform of a Resonance Term (Sec. 2.8)**

Let $f(t) = t \sin \omega t$. Then $f(0) = 0$, $f'(t) = \sin \omega t + \omega t \cos \omega t$, $f'(0) = 0$, $f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}. \quad \blacksquare$$

EXAMPLE 2 Formulas 7 and 8 in Table 6.1, Sec. 6.1

This is a third derivation of $\mathcal{L}(\cos \omega t)$ and $\mathcal{L}(\sin \omega t)$; cf. Example 4 in Sec. 6.1. Let $f(t) = \cos \omega t$. Then $f(0) = 1$, $f'(0) = 0$, $f''(t) = -\omega^2 \cos \omega t$. From this and (2) we obtain

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f). \qquad \text{By algebra,} \qquad \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Similarly, let $g = \sin \omega t$. Then $g(0) = 0$, $g' = \omega \cos \omega t$. From this and (1) we obtain

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos \omega t). \qquad \text{Hence} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s}\mathcal{L}(\cos \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

■

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

THEOREM 3 Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\text{(4)} \qquad \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

PROOF Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right| \leqq \int_0^t |f(\tau)|\, d\tau \leqq M \int_0^t e^{k\tau}\, d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k}e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0) = 0$ (the integral from 0 to 0 is zero)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. ■

EXAMPLE 3 Application of Theorem 3: Formulas 19 and 20 in the Table of Sec. 6.9

Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2 + \omega^2)}$ and $\dfrac{1}{s^2(s^2 + \omega^2)}$.

Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega\tau}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This is formula 19 in Sec. 6.9. Integrating this result again and using (4) as before, we obtain formula 20 in Sec. 6.9:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}.$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction. ■

Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1 \tag{5}$$

where a and b are constant. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained. In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming (5) by means of (1) and (2), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + as + b$ and use the so-called **transfer function**

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}. \tag{6}$$

(Q is often denoted by H, but we need H much more frequently for other purposes.) This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s). \tag{7}$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

and this explains the name of Q. Note that ***Q depends neither on $r(t)$ nor on the initial conditions*** (but only on a and b).

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$. We reduce (7) (usually by ***partial fractions*** as in calculus) to a sum of terms whose inverses can be found from the tables (e.g., in Sec. 6.1 or Sec. 6.9) or by a CAS, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ *of* (5).

EXAMPLE 4 Initial Value Problem: The Basic Laplace Steps

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. *Step 1.* From (2) and Table 6.1 we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}\,Q = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}.$$

Simplification and **partial fraction expansion** gives

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

The diagram in Fig. 115 summarizes our approach. ■

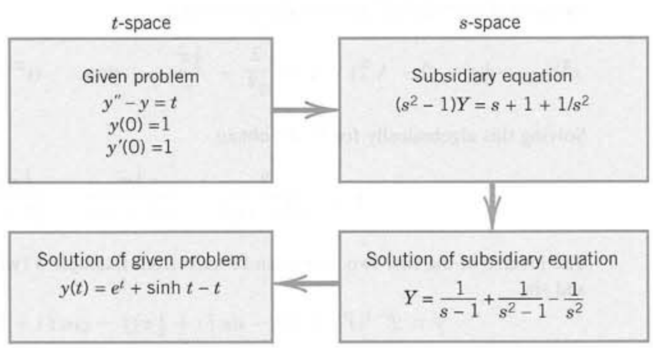


Fig. 115. Laplace transform method

EXAMPLE 5 Comparison with the Usual Method

Solve the initial value problem

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Solution. From (1) and (2) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{thus} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

The solution is

$$Y = \frac{0.16(s + 1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}.$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 6.1 we obtain

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16 \cos\sqrt{\frac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin\sqrt{\frac{35}{4}}\,t\right)$$

$$= e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t).$$

This agrees with Example 2, Case (III) in Sec. 2.4. The work was less. ■

Advantages of the Laplace Method

1. *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.* See Example 4.
2. *Initial values are automatically taken care of.* See Examples 4 and 5.
3. *Complicated inputs* $r(t)$ (right sides of linear ODEs) *can be handled very efficiently,* as we show in the next sections.

EXAMPLE 6 Shifted Data Problems

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s\cdot\tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned}\tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) &= 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}.\end{aligned}$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin\tilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t. \qquad \blacksquare$$

PROBLEM SET 6.2

1–8 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

1. te^{kt}

2. $t \cos 5t$

3. $\sin^2 \omega t$

4. $\cos^2 \pi t$

5. $\sinh^2 at$

6. $\cosh^2 \frac{1}{2}t$

7. $t \sin \frac{1}{2}\pi t$

8. $\sin^4 t$ (Use Prob. 3.)

9. (Derivation by different methods) It is typical that various transforms can be obtained by several methods. Show this for Prob. 1. Show it for $\mathcal{L}(\cos^2 \frac{1}{2}t)$ (a) by expressing $\cos^2 \frac{1}{2}t$ in terms of $\cos t$, (b) by using Prob. 3.

10–24 INITIAL VALUE PROBLEMS

Solve the following initial value problems by the Laplace transform. (If necessary, use partial fraction expansion as in Example 4. Show all details.)

10. $y' + 4y = 0, \quad y(0) = 2.8$

11. $y' + \frac{1}{2}y = 17 \sin 2t, \quad y(0) = -1$

12. $y'' - y' - 6y = 0, \quad y(0) = 6, \quad y'(0) = 13$

13. $y'' - \frac{1}{4}y = 0, \quad y(0) = 4, \quad y'(0) = 0$

14. $y'' - 4y' + 4y = 0, \quad y(0) = 2.1,$ $y'(0) = 3.9$

15. $y'' + 2y' + 2y = 0, \quad y(0) = 1,$ $y'(0) = -3$

16. $y'' + ky' - 2k^2y = 0, \quad y(0) = 2,$ $y'(0) = 2k$

17. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5,$ $y'(0) = -10$

18. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1,$ $y'(0) = 31.5$

20. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2,$ $y'(0) = 6.2$

21. **(Shifted data)** $y' - 6y = 0, \quad y(2) = 4$

22. $y'' - 2y' - 3y = 0, \quad y(1) = -3,$ $y'(1) = -17$

23. $y'' + 3y' - 4y = 6e^{2t-2}, \quad y(1) = 4,$ $y'(1) = 5$

24. $y'' + 2y' + 5y = 50t - 150, \quad y(3) = -4,$ $y'(3) = 14$

25. PROJECT. Comments on Sec. 6.2. **(a)** Give reasons why Theorems 1 and 2 are more important than Theorem 3.

(b) Extend Theorem 1 by showing that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at some $t = a\ (> 0)$, the other conditions remaining as in Theorem 1, then (see Fig. 116)

$$\text{(1*)} \quad \mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a + 0) - f(a - 0)]e^{-as}.$$

(c) Verify (1*) for $f(t) = e^{-t}$ if $0 < t < 1$ and 0 if $t > 1$.

(d) Verify (1*) for two more complicated functions of your choice.

(e) Compare the Laplace transform of solving ODEs with the method in Chap. 2. Give examples of your own to illustrate the advantages of the present method (to the extent we have seen them so far).

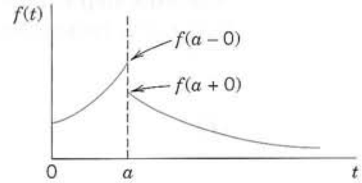


Fig. 116. Formula (1*)

26. PROJECT. Further Results by Differentiation. Proceeding as in Example 1, obtain

$$\text{(a)} \quad \mathcal{L}(t \cos \omega t) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

and from this and Example 1: (b) formula 21, (c) 22, (d) 23 in Sec. 6.9,

$$\text{(e)} \quad \mathcal{L}(t \cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2},$$

$$\text{(f)} \quad \mathcal{L}(t \sinh at) = \frac{2as}{(s^2 - a^2)^2}.$$

27–34 OBTAINING TRANSFORMS BY INTEGRATION

Using Theorem 3, find $f(t)$ if $\mathcal{L}(f)$ equals:

27. $\dfrac{1}{s^2 + s/2}$

28. $\dfrac{10}{s^3 - \pi s^2}$

29. $\dfrac{1}{s^3 - ks^2}$

30. $\dfrac{1}{s^4 + s^2}$

31. $\dfrac{5}{s^3 - 5s}$

32. $\dfrac{2}{s^3 + 9s}$

33. $\dfrac{1}{s^4 - 4s^2}$

34. $\dfrac{1}{s^4 + \pi^2 s^2}$

35. **(Partial fractions)** Solve Probs. 27, 29, and 31 by using partial fractions.

6.3 Unit Step Function. t-Shifting

This section and the next one are extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach of Chap. 2. The reason is that we shall introduce two auxiliary functions, the *unit step function* or *Heaviside function* $u(t - a)$ (below) and *Dirac's delta* $\delta(t - a)$ (in Sec. 6.4). These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (hammerblows, for example).

Unit Step Function (Heaviside Function) $u(t - a)$

The **unit step function** or **Heaviside function** $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0). \tag{1}$$

Figure 117 shows the special case $u(t)$, which has its jump at zero, and Fig. 118 the general case $u(t - a)$ for an arbitrary positive a. (For Heaviside see Sec. 6.1.)

The transform of $u(t - a)$ follows directly from the defining integral in Sec. 6.1,

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\cdot 1\,dt = -\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty};$$

here the integration begins at $t = a$ ($\geqq 0$) because $u(t - a)$ is 0 for $t < a$. Hence

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s > 0). \tag{2}$$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on." Multiplying functions $f(t)$ with $u(t - a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 119 and 120. In Fig. 119 the given function is shown in (A). In (B) it is switched off between $t = 0$ and $t = 2$ (because $u(t - 2) = 0$ when $t < 2$) and is switched on beginning at $t = 2$. In (C) it is shifted to the right by 2 units, say, for instance, by 2 secs, so that it begins 2 secs later in the same fashion as before. More generally we have the following.

Let $f(t) = 0$ for all negative t. Then $f(t - a)u(t - a)$ with $a > 0$ is $f(t)$ ***shifted*** *(translated) to the right by the amount a.*

Figure 120 shows the effect of many unit step functions, three of them in (A) and infinitely many in (B) when continued periodically to the right; this is the effect of a rectifier that clips off the negative half-waves of a sinuosidal voltage. CAUTION! Make sure that you fully understand these figures, in particular the difference between parts (B) and (C) of Figure 119. Figure 119(C) will be applied next.

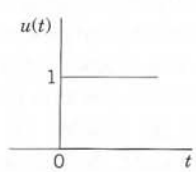


Fig. 117. Unit step function $u(t)$

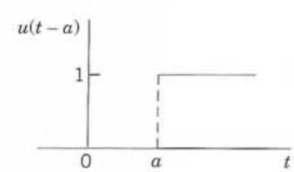
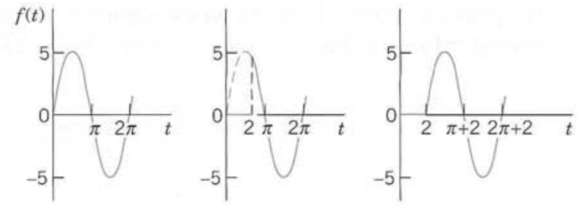


Fig. 118. Unit step function $u(t - a)$



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Fig. 119. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin (\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 120. Use of many unit step functions.

Time Shifting (*t*-Shifting): Replacing *t* by *t* − *a* in *f*(*t*)

The first shifting theorem ("*s*-shifting") in Sec. 6.1 concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s - a) = \mathcal{L}\{e^{at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t - a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

THEOREM 1

Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the **"shifted function"**

(3) $$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

(4) $$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

(4*) $$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. In Fig. 119, the transform of $5 \sin t$ is $F(s) = 5/(s^2 + 1)$, hence the shifted function $5 \sin (t - 2)\, u(t - 2)$ shown in Fig. 119(C) has the transform

$$e^{-2s}F(s) = 5e^{-2s}/(s^2 + 1).$$

PROOF We prove Theorem 1. In (4) on the right we use the definition of the Laplace transform, writing τ for t (to have t available later). Then, taking e^{-as} inside the integral, we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau.$$

Substituting $\tau + a = t$, thus $\tau = t - a$, $d\tau = dt$, in the integral (CAUTION, the lower limit changes!), we obtain

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt.$$

To make the right side into a Laplace transform, we must have an integral from 0 to ∞, not from a to ∞. But this is easy. We multiply the integrand by $u(t-a)$. Then for t from 0 to a the integrand is 0, and we can write, with $\widetilde{f}$ as in (3),

$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt = \int_0^\infty e^{-st}\widetilde{f}(t)\,dt.$$

(Do you now see why $u(t-a)$ appears?) This integral is the left side of (4), the Laplace transform of $\widetilde{f}(t)$ in (3). This completes the proof. ■

EXAMPLE 1 Application of Theorem 1. Use of Unit Step Functions

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \tfrac{1}{2}t^2 & \text{if } 1 < t < \tfrac{1}{2}\pi \\ \cos t & \text{if } \quad t > \tfrac{1}{2}\pi. \end{cases} \qquad \text{(Fig. 121)}$$

Solution. *Step 1.* In terms of unit step functions,

$$f(t) = 2(1 - u(t-1)) + \tfrac{1}{2}t^2(u(t-1) - u(t - \tfrac{1}{2}\pi)) + (\cos t)u(t - \tfrac{1}{2}\pi).$$

Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2. To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1 - u(t-1))$ remains as it is and gives the transform $2(1 - e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t - \frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t - \frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\textbf{(4**)} \qquad \mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

(4**) follows from (4) by writing $f(t-a) = g(t)$, hence $f(t) = g(t+a)$ and then again writing f for g. Thus,

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2 + t + \frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

as before. Similarly for $\mathcal{L}\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\}$. Finally, by (4**),

$$\mathcal{L}\left\{\cos t\, u\left(t - \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t + \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}. \qquad \blacksquare$$

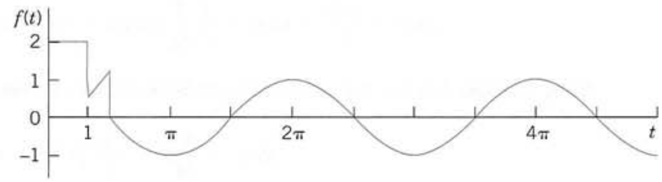


Fig. 121. $f(t)$ in Example 1

EXAMPLE 2 **Application of Both Shifting Theorems. Inverse Transform**

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Solution. Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} because $1/s^2$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the first shifting theorem in Sec. 6.1. Hence by the second shifting theorem (t-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))\,u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))\,u(t-2) + (t-3)e^{-2(t-3)}\,u(t-3).$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so that the second and third terms cancel each other when $t > 2$. Hence we obtain $f(t) = 0$ if $0 < t < 1$, $-(\sin \pi t)/\pi$ if $1 < t < 2$, 0 if $2 < t < 3$, and $(t-3)e^{-2(t-3)}$ if $t > 3$. See Fig. 122. $\blacksquare$

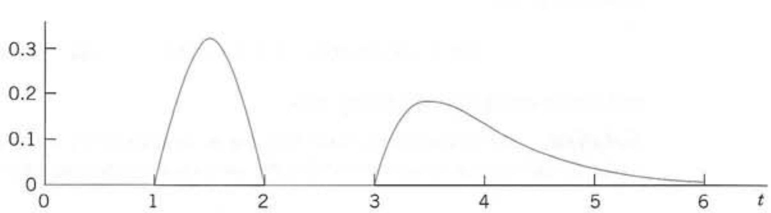


Fig. 122. $f(t)$ in Example 2

EXAMPLE 3 **Response of an *RC*-Circuit to a Single Rectangular Wave**

Find the current $i(t)$ in the RC-circuit in Fig. 123 if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the wave is applied.

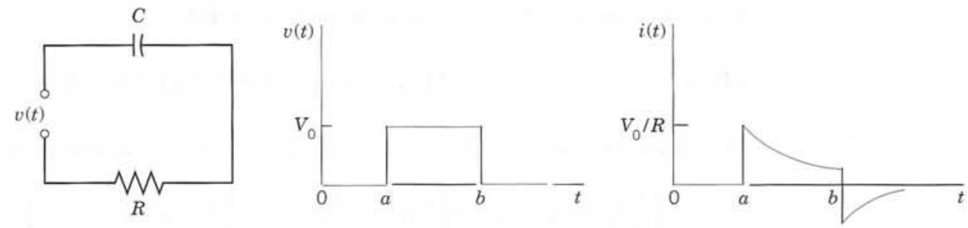


Fig. 123. *RC*-circuit, electromotive force *v*(*t*), and current in Example 3

Solution. The input is $V_0[u(t-a)-u(t-b)]$. Hence the circuit is modeled by the integro-differential equation (see Sec. 2.9 and Fig. 123)

$$Ri(t)+\frac{q(t)}{C}=Ri(t)+\frac{1}{C}\int_0^t i(\tau)\,d\tau=v(t)=V_0[u(t-a)-u(t-b)].$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s)+\frac{I(s)}{sC}=\frac{V_0}{s}\,[e^{-as}-e^{-bs}].$$

Solving this equation algebraically for $I(s)$, we get

$$I(s)=F(s)(e^{-as}-e^{-bs})\qquad\text{where}\qquad F(s)=\frac{V_0/R}{s+1/(RC)}\qquad\text{and}\qquad \mathcal{L}^{-1}(F)=\frac{V_0}{R}\,e^{-t/(RC)},$$

the last expression being obtained from Table 6.1 in Sec. 6.1. Hence Theorem 1 yields the solution (Fig. 123)

$$i(t)=\mathcal{L}^{-1}(I)=\mathcal{L}^{-1}\{e^{-as}F(s)\}-\mathcal{L}^{-1}\{e^{-bs}F(s)\}=\frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a)-e^{-(t-b)/(RC)}u(t-b)\right];$$

that is, $i(t)=0$ if $t<a$, and

$$i(t)=\begin{cases}K_1e^{-t/(RC)} & \text{if } a<t<b\\ (K_1-K_2)e^{-t/(RC)} & \text{if } a>b\end{cases}$$

where $K_1=V_0e^{a/(RC)}/R$ and $K_2=V_0e^{b/(RC)}/R$. ■

EXAMPLE 4 Response of an *RLC*-Circuit to a Sinusoidal Input Acting Over a Time Interval

Find the response (the current) of the *RLC*-circuit in Fig. 124, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t)=100\sin 400t\quad\text{if } 0<t<2\pi\qquad\text{and}\qquad E(t)=0\text{ if } t>2\pi$$

and current and charge are initially zero.

Solution. The electromotive force $E(t)$ can be represented by $(100\sin 400t)(1-u(t-2\pi))$. Hence the model for the current $i(t)$ in the circuit is the integro-differential equation (see Sec. 2.9)

$$0.1i'+11i+100\int_0^t i(\tau)\,d\tau=(100\sin 400t)(1-u(t-2\pi)),\qquad i(0)=0,\quad i'(0)=0.$$

From Theorems 2 and 3 in Sec. 6.2 we obtain the subsidiary equation for $I(s)=\mathcal{L}(i)$

$$0.1sI+11I+100\,\frac{I}{s}=\frac{100\cdot 400s}{s^2+400^2}\left(\frac{1}{s}-\frac{e^{-2\pi s}}{s}\right).$$

Solving it algebraically and noting that $s^2 + 110s + 1000 = (s + 10)(s + 100)$, we obtain

$$I(s) = \frac{1000 \cdot 400}{(s + 10)(s + 100)} \left(\frac{s}{s^2 + 400^2} - \frac{se^{-2\pi s}}{s^2 + 400^2} \right).$$

For the first term in the parentheses $(\cdots)$ times the factor in front of them we use the partial fraction expansion

$$\frac{400\,000s}{(s + 10)(s + 100)(s^2 + 400^2)} = \frac{A}{s + 10} + \frac{B}{s + 100} + \frac{Ds + K}{s^2 + 400^2}.$$

Now determine A, B, D, K by your favorite method or by a CAS or as follows. Multiplication by the common denominator gives

$$400\,000s = A(s + 100)(s^2 + 400^2) + B(s + 10)(s^2 + 400^2) + (Ds + K)(s + 10)(s + 100).$$

We set $s = -10$ and -100 and then equate the sums of the s^3 and s^2 terms to zero, obtaining (all values rounded)

$$\begin{array}{lrl} (s = -10) & -4\,000\,000 = 90(10^2 + 400^2)A, & A = -0.27760 \\ (s = -100) & -40\,000\,000 = -90(100^2 + 400^2)B, & B = 2.6144 \\ (s^3\text{-terms}) & 0 = A + B + D, & D = -2.3368 \\ (s^2\text{-terms}) & 0 = 100A + 10B + 110D + K, & K = 258.66. \end{array}$$

Since $K = 258.66 = 0.6467 \cdot 400$, we thus obtain for the first term I_1 in $I = I_1 - I_2$

$$I_1 = -\frac{0.2776}{s + 10} + \frac{2.6144}{s + 100} - \frac{2.3368s}{s^2 + 400^2} + \frac{0.6467 \cdot 400}{s^2 + 400^2}.$$

From Table 6.1 in Sec. 6.1 we see that its inverse is

$$i_1(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

This is the current $i(t)$ when $0 < t < 2\pi$. It agrees for $0 < t < 2\pi$ with that in Example 1 of Sec. 2.9 (except for notation), which concerned the same RLC-circuit. Its graph in Fig. 62 in Sec. 2.9 shows that the exponential terms decrease very rapidly. Note that the present amount of work was substantially less.

The second term I_1 of I differs from the first term by the factor $e^{-2\pi s}$. Since $\cos 400(t - 2\pi) = \cos 400t$ and $\sin 400(t - 2\pi) = \sin 400t$, the second shifting theorem (Theorem 1) gives the inverse $i_2(t) = 0$ if $0 < t < 2\pi$, and for $> 2\pi$ it gives

$$i_2(t) = -0.2776e^{-10(t-2\pi)} + 2.6144e^{-100(t-2\pi)} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Hence in $i(t)$ the cosine and sine terms cancel, and the current for $t > 2\pi$ is

$$i(t) = -0.2776(e^{-10t} - e^{-10(t-2\pi)}) + 2.6144(e^{-100t} - e^{-100(t-2\pi)}).$$

It goes to zero very rapidly, practically within 0.5 sec. ■

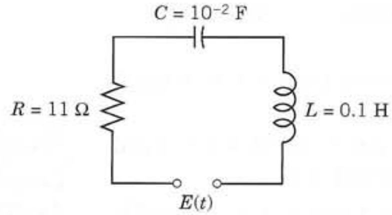


Fig. 124. RLC-circuit in Example 4

PROBLEM SET 6.3

1. **WRITING PROJECT. Shifting Theorem.** Explain and compare the different roles of the two shifting theorems, using your own formulations and examples.

2–13 UNIT STEP FUNCTION AND SECOND SHIFTING THEOREM

Sketch or graph the given function (which is assumed to be zero outside the given interval). Represent it using unit step functions. Find its transform. Show the details of your work.

2. t $(0 < t < 1)$
3. e^t $(0 < t < 2)$
4. $\sin 3t$ $(0 < t < \pi)$
5. t^2 $(1 < t < 2)$
6. t^2 $(t > 3)$
7. $\cos \pi t$ $(1 < t < 4)$
8. $1 - e^{-t}$ $(0 < t < \pi)$
9. t $(5 < t < 10)$
10. $\sin \omega t$ $(t > 6\pi/\omega)$
11. $20 \cos \pi t$ $(3 < t < 6)$
12. $\sinh t$ $(0 < t < 2)$
13. $e^{\pi t}$ $(2 < t < 4)$

14–22 INVERSE TRANSFORMS BY THE SECOND SHIFTING THEOREM

Find and sketch or graph $f(t)$ if $\mathcal{L}(f)$ equals:

14. $se^{-s}/(s^2 + \omega^2)$
15. e^{-4s}/s^2
16. $s^{-2} - (s^{-2} + s^{-1})e^{-s}$
17. $(e^{-2\pi s} - e^{-8\pi s})/(s^2 + 1)$
18. $e^{-\pi s}/(s^2 + 2s + 2)$
19. e^{-2s}/s^5
20. $(1 - e^{-s+k})/(s - k)$
21. $se^{-3s}/(s^2 - 4)$
22. $2.5(e^{-3.8s} - e^{-2.6s})/s$

23–34 INITIAL VALUE PROBLEMS, SOME WITH DISCONTINUOUS INPUTS

Using the Laplace transform and showing the details, solve:

23. $y'' + 2y' + 2y = 0,$ $y(0) = 0,$ $y'(0) = 1$
24. $9y'' - 6y' + y = 0,$ $y(0) = 3,$ $y'(0) = 1$
25. $y'' + 4y' + 13y = 145 \cos 2t,$ $y(0) = 10,$ $y'(0) = 14$
26. $y'' + 10y' + 24y = 144t^2,$ $y(0) = \frac{19}{12},$ $y'(0) = -5$
27. $y'' + 9y = r(t),$ $r(t) = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi;$ $y(0) = 0,$ $y'(0) = 4$
28. $y'' + 3y' + 2y = r(t),$ $r(t) = 1$ if $0 < t < 1$ and 0 if $t > 1;$ $y(0) = 0,$ $y'(0) = 0$
29. $y'' + y = r(t),$ $r(t) = t$ if $0 < t < 1$ and 0 if $t > 1;$ $y(0) = y'(0) = 0$
30. $y'' - 16y = r(t),$ $r(t) = 48e^{2t}$ if $0 < t < 4$ and 0 if $t > 4;$ $y(0) = 3,$ $y'(0) = -4$
31. $y'' + y' - 2y = r(t),$ $r(t) = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi;$ $y(0) = 1,$ $y'(0) = 0$
32. $y'' + 8y' + 15y = r(t),$ $r(t) = 35e^{2t}$ if $0 < t < 2$ and 0 if $t > 2;$ $y(0) = 3,$ $y'(0) = -8$
33. **(Shifted data)** $y'' + 4y = 8t^2$ if $0 < t < 5$ and 0 if $t > 5;$ $y(1) = 1 + \cos 2,$ $y'(1) = 4 - 2 \sin 2$
34. $y'' + 2y' + 5y = 10 \sin t$ if $0 < t < 2\pi$ and 0 if $t > 2\pi;$ $y(\pi) = 1,$ $y'(\pi) = 2e^{-\pi} - 2$

MODELS OF ELECTRIC CIRCUITS

35. **(Discharge)** Using the Laplace transform, find the charge $q(t)$ on the capacitor of capacitance C in Fig. 125 if the capacitor is charged so that its potential is V_0 and the switch is closed at $t = 0$.

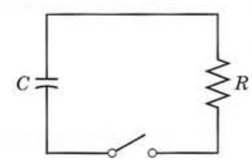


Fig. 125. Problem 35

36–38 RC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 126 with $R = 10\ \Omega$ and $C = 10^{-2}$ F, where the current at $t = 0$ is assumed to be zero, and:

36. $v(t) = 100$ V if $0.5 < t < 0.6$ and 0 otherwise. Why does $i(t)$ have jumps?
37. $v = 0$ if $t < 2$ and $100(t - 2)$ V if $t > 2$
38. $v = 0$ if $t < 4$ and $14 \cdot 10^6 e^{-3t}$ V if $t > 4$

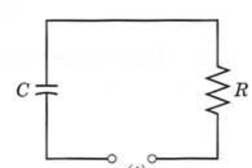


Fig. 126. Problems 36–38

39–41 RL-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 127, assuming $i(0) = 0$ and:

39. $R = 10\ \Omega$, $L = 0.5$ H, $v = 200t$ V if $0 < t < 2$ and 0 if $t > 2$

40. $R = 1\ \text{k}\Omega\ (= 1000\ \Omega)$, $L = 1$ H, $v = 0$ if $0 < t < \pi$, and $40 \sin t$ V if $t > \pi$

41. $R = 25\ \Omega$, $L = 0.1$ H, $v = 490e^{-5t}$ V if $0 < t < 1$ and 0 if $t > 1$

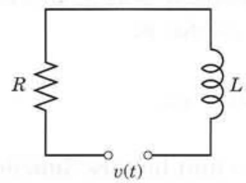


Fig. 127. Problems 39–41

42–44 LC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 128, assuming zero initial current and charge on the capacitor and:

42. $L = 1$ H, $C = 0.25$ F, $v = 200(t - \frac{1}{3}t^3)$ V if $0 < t < 1$ and 0 if $t > 1$

43. $L = 1$ H, $C = 10^{-2}$ F, $v = -9900 \cos t$ V if $\pi < t < 3\pi$ and 0 otherwise

44. $L = 0.5$ H, $C = 0.05$ F, $v = 78 \sin t$ V if $0 < t < \pi$ and 0 if $t > \pi$

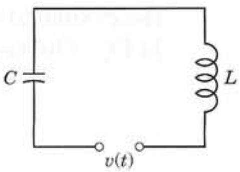


Fig. 128. Problems 42–44

45–47 RLC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 129, assuming zero initial current and charge and:

45. $R = 2\ \Omega$, $L = 1$ H, $C = 0.5$ F, $v(t) = 1$ kV if $0 < t < 2$ and 0 if $t > 2$

46. $R = 4\ \Omega$, $L = 1$ H, $C = 0.05$ F, $v = 34e^{-t}$ V if $0 < t < 4$ and 0 if $t > 4$

47. $R = 2\ \Omega$, $L = 1$ H, $C = 0.1$ F, $v = 255 \sin t$ V if $0 < t < 2\pi$ and 0 if $t > 2\pi$

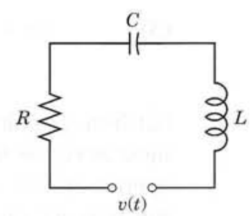


Fig. 129. Problems 45–47

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

Phenomena of an impulsive nature, such as the action of forces or voltages over short intervals of time, arise in various applications, for instance, if a mechanical system is hit by a hammerblow, an airplane makes a "hard" landing, a ship is hit by a single high wave, or we hit a tennisball by a racket, and so on. Our goal is to show how such problems are modeled by "Dirac's delta function" and can be solved very efficiently by the Laplace transform.

To model situations of that type, we consider the function

$$(1) \qquad f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a+k \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig. 130)}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \leqq t \leqq a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 130 has area 1, the impulse of f_k in (1) is

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1. \tag{2}$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ $(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

$\delta(t-a)$ is called the **Dirac delta function**[2] or the **unit impulse function.**

$\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called *generalized function.*[2] To see this, we note that the impulse I_k of f_k is 1, so that from (1) and (2) by taking the limit as $k \to 0$ we obtain

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\,dt = 1, \tag{3}$$

but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a *continuous* function $g(t)$ one uses the property [often called the **sifting property** of $\delta(t-a)$, not to be confused with *shifting*]

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a) \tag{4}$$

which is plausible by (2).

To obtain the Laplace transform of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}\,[u(t-a) - u(t-(a+k))]$$

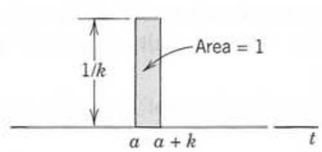


Fig. 130. The function $f_k(t-a)$ in (1)

[2]PAUL DIRAC (1902–1984), English physicist, was awarded the Nobel Prize [jointly with the Austrian ERWIN SCHRÖDINGER (1887–1961)] in 1933 for his work in quantum mechanics.

Generalized functions are also called **distributions.** Their theory was created in 1936 by the Russian mathematician SERGEI L'VOVICH SOBOLEV (1908–1989), and in 1945, under wider aspects, by the French mathematician LAURENT SCHWARTZ (1915–2002).

and take the transform [see (2)]

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}.$$

We now take the limit as $k \to 0$. By l'Hôpital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t-a)$ by this limit, that is,

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}. \tag{5}$$

The unit step and unit impulse functions can now be used on the right side of ODEs modeling mechanical or electrical systems, as we illustrate next.

EXAMPLE 1 **Mass–Spring System Under a Square Wave**

Determine the response of the damped mass–spring system (see Sec. 2.8) under a square wave, modeled by (see Fig. 131)

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution. From (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \frac{1}{s}\left(e^{-s} - e^{-2s}\right). \qquad \text{Solution} \qquad Y(s) = \frac{1}{s(s^2+3s+2)}\left(e^{-s} - e^{-2s}\right).$$

Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

Therefore, by Theorem 1 in Sec. 6.3 (t-shifting) we obtain the square-wave response shown in Fig. 131,

$$\begin{aligned} y &= \mathcal{L}^{-1}\left(F(s)e^{-s} - F(s)e^{-2s}\right) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-1)} - \tfrac{1}{2}e^{-2(t-2)} & (t > 2). \end{cases} \end{aligned}$$

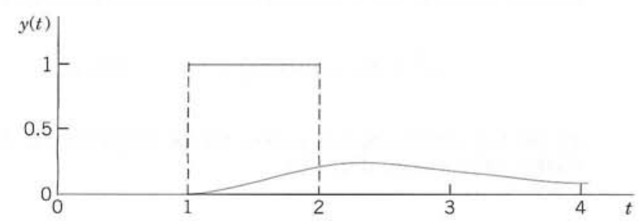


Fig. 131. Square wave and response in Example 1

EXAMPLE 2 Hammerblow Response of a Mass–Spring System

Find the response of the system in Example 1 with the square wave replaced by a unit impulse at time $t = 1$.

Solution. We now have the ODE and the subsidiary equation

$$y'' + 3y' + 2y = \delta(t-1), \qquad \text{and} \qquad (s^2 + 3s + 2)Y = e^{-s}.$$

Solving algebraically gives

$$Y(s) = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

By Theorem 1 the inverse is

$$y(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } \quad t > 1. \end{cases}$$

$y(t)$ is shown in Fig. 132. Can you imagine how Fig. 131 approaches Fig. 132 as the wave becomes shorter and shorter, the area of the rectangle remaining 1? ■

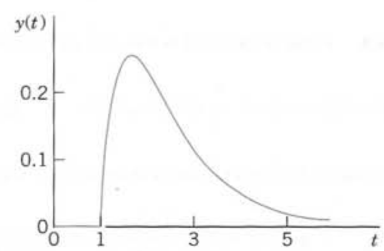


Fig. 132. Response to a hammerblow in Example 2

EXAMPLE 3 Four-Terminal *RLC*-Network

Find the output voltage response in Fig. 133 if $R = 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

Solution. To understand what is going on, note that the network is an *RLC*-circuit to which two wires at A and B are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

$$Li' + Ri + \frac{q}{C} = Lq'' + Rq' + \frac{q}{C} = q'' + 20q' + 10\,000q = \delta(t).$$

From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

$$(s^2 + 20s + 10\,000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from Q damped oscillations for q and v; rounding $9900 \approx 99.50^2$, we get (Fig. 133)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50}e^{-10t}\sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t}\sin 99.50t. \qquad ■$$

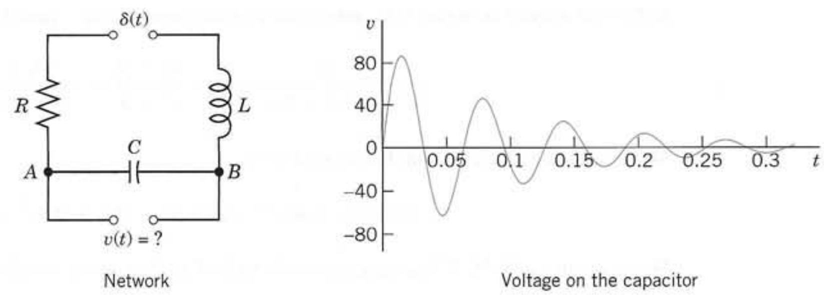


Fig. 133. Network and output voltage in Example 3

More on Partial Fractions

We have seen that the solution Y of a subsidiary equation usually appears as a quotient of polynomials $Y(s) = F(s)/G(s)$, so that a partial fraction representation leads to a sum of expressions whose inverses we can obtain from a table, aided by the first shifting theorem (Sec. 6.1). These representations are sometimes called **Heaviside expansions.**

An *unrepeated factor* $s - a$ in $G(s)$ requires a single partial fraction $A/(s - a)$. See Examples 1 and 2 on pp. 243, 244. *Repeated real factors* $(s - a)^2$, $(s - a)^3$, etc., require partial fractions

$$\frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \qquad \frac{A_3}{(s-a)^3} + \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \quad \text{etc.,}$$

The inverses are $(A_2 t + A_1)e^{at}$, $(\frac{1}{2}A_3 t^2 + A_2 t + A_1)e^{at}$, etc.

Unrepeated complex factors $(s - a)(s - \bar{a})$, $a = \alpha + i\beta$, $\bar{a} = \alpha - i\beta$, require a partial fraction $(As + B)/[(s - \alpha)^2 + \beta^2]$. For an application, see Example 4 in Sec. 6.3. A further one is the following.

EXAMPLE 4 Unrepeated Complex Factors. Damped Forced Vibrations

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval (Fig. 134),

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

Solution. From Table 6.1, (1), (2) in Sec. 6.2, and the second shifting theorem in Sec. 6.3, we obtain the subsidiary equation

$$(s^2 Y - s + 5) + 2(sY - 1) + 2Y = 10\,\frac{2}{s^2 + 4}\,(1 - e^{-\pi s}).$$

We collect the Y-terms, $(s^2 + 2s + 2)Y$, take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$Y = \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}. \tag{6}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{7}$$

In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}.$$

Multiplication by the common denominator gives

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

$$\text{(a)}\ [s^3]:\ 0 = A + M \qquad \text{(b)}\ [s^2]:\ 0 = 2A + B + N$$

$$\text{(c)}\ [s]:\ 0 = 2A + 2B + 4M \qquad \text{(d)}\ [s^0]:\ 20 = 2B + 4N.$$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (6) has the representation

$$\text{(8)}\quad \frac{-2s-2}{s^2+4} + \frac{2(s+1)+6-2}{(s+1)^2+1}. \quad \text{Inverse transform:}\quad -2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t).$$

The sum of this and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$\text{(9)}\qquad y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi.$$

In the second fraction in (6) taken with the minus sign we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform

$$+2\cos(2t-2\pi) + \sin(2t-2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$
$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t).$$

The sum of this and (9) is the solution for $t > \pi$,

$$\text{(10)}\qquad y(t) = e^{-t}[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi.$$

Figure 134 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

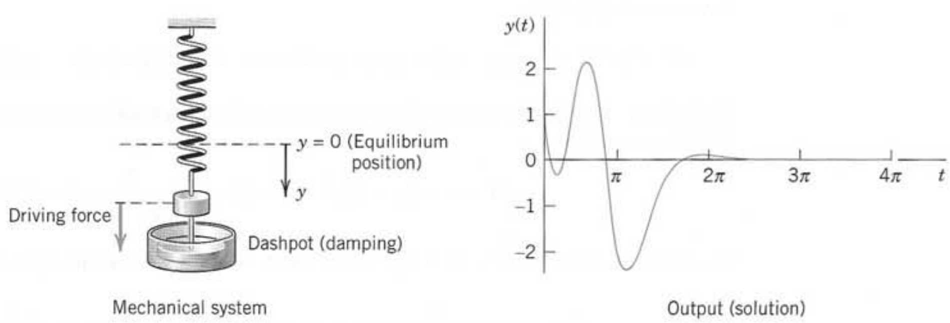


Fig. 134. Example 4

The case of repeated complex factors $[(s-a)(s-\bar{a})]^2$, which is important in connection with resonance, will be handled by "convolution" in the next section.

PROBLEM SET 6.4

1–12 EFFECT OF DELTA FUNCTION ON VIBRATING SYSTEMS

Showing the details, find, graph, and discuss the solution.

1. $y'' + y = \delta(t - 2\pi)$, $y(0) = 10$, $y'(0) = 0$
2. $y'' + 2y' + 2y = e^{-t} + 5\,\delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$
3. $y'' - y = 10\delta(t - \frac{1}{2}) - 100\delta(t - 1)$, $y(0) = 10$, $y'(0) = 1$
4. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1))$, $y(0) = 1$, $y'(0) = -1$
5. $y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\delta(t - 10)$, $y(0) = 0$, $y'(0) = 1$
6. $y'' + 2y' - 3y = 100\delta(t - 2) + 100\delta(t - 3)$, $y(0) = 1$, $y'(0) = 0$
7. $y'' + 2y' + 10y = 10[1 - u(t - 4)] - 10\delta(t - 5)$, $y(0) = 1$, $y'(0) = 1$
8. $y'' + 5y' + 6y = \delta(t - \frac{1}{2}\pi) + u(t - \pi)\cos t$, $y(0) = 0$, $y'(0) = 0$
9. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$
10. $y'' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$. (Compare with Prob. 9.)
11. $y'' + 3y' - 4y = 2e^t - 8e^2\delta(t - 2)$, $y(0) = 2$, $y'(0) = 0$
12. $y'' + y = -2\sin t + 10\delta(t - \pi)$, $y(0) = 0$, $y'(0) = 1$

13. **CAS PROJECT. Effect of Damping.** Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = r(t)$$

with $r(t)$ involving a δ-function. **(a)** Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

14. **CAS PROJECT. Limit of a Rectangular Wave. Effects of Impulse.**

(a) In Example 1, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 132. *Hint:* If your CAS gives no solution for the differential equation. involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

15. **PROJECT. Heaviside Formulas. (a)** Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots + \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)} \right], \quad k = 1, \cdots, m - 1.$$

16. **TEAM PROJECT. Laplace Transform of Periodic Functions**

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of sin ωt in Fig. 135 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 136.)

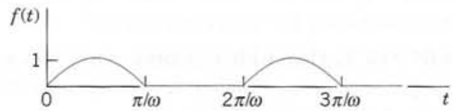


Fig. 135. Half-wave rectification

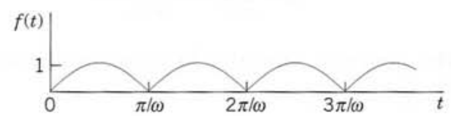


Fig. 136. Full-wave rectification

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of sin ωt is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 137.

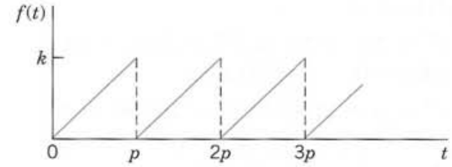


Fig. 137. Saw-tooth wave

(e) Staircase function. Find the Laplace transform of the staircase function in Fig. 138 by noting that it is the difference of kt/p and the function in (d).

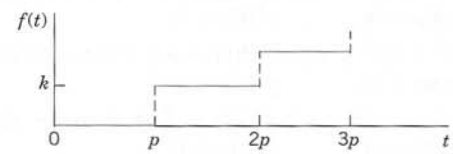


Fig. 138. Staircase function

6.5 Convolution. Integral Equations

Convolution has to do with the multiplication of transforms. The situation is as follows. *Addition* of transforms provides no problem; we know that $\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$. Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product $\mathcal{L}(f)\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false. *The transform of a product is generally different from the product of the transforms of the factors,*

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g) \qquad \text{in general.}$$

To see this take $f = e^t$ and $g = 1$. Then $fg = e^t$, $\mathcal{L}(fg) = 1/(s - 1)$, but $\mathcal{L}(f) = 1/(s - 1)$ and $\mathcal{L}(1) = 1/s$ give $\mathcal{L}(f)\mathcal{L}(g) = 1/(s^2 - s)$.

According to the next theorem, the correct answer is that $\mathcal{L}(f)\mathcal{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f * g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau. \tag{1}$$

THEOREM 1 **Convolution Theorem**

If two functions f and g satisfy the assumption in the existence theorem in Sec. 6.1, so that their transforms F and G exist, the product $H = FG$ is the transform of h given by (1). (Proof after Example 2.)

EXAMPLE 1 **Convolution**

Let $H(s) = 1/[(s - a)s]$. Find $h(t)$.

Solution. $1/(s - a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t - \tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau = \frac{1}{a}(e^{at} - 1).$$

To check, calculate

$$H(s) = \mathscr{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s - a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s - a} \cdot \frac{1}{s} = \mathscr{L}(e^{at})\,\mathscr{L}(1).$$

EXAMPLE 2 **Convolution**

Let $H(s) = 1/(s^2 + \omega^2)^2$. Find $h(t)$.

Solution. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$. Hence from (1) and the trigonometric formula (11) in App. 3.1 with $x = \frac{1}{2}(\omega t + \omega\tau)$ and $y = \frac{1}{2}(\omega t - \omega\tau)$ we obtain

$$\begin{aligned} h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} &= \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t - \tau)\, d\tau \\ &= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos \omega\tau]\, d\tau \\ &= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t} \\ &= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right] \end{aligned}$$

in agreement with formula 21 in the table in Sec. 6.9.

PROOF We prove the Convolution Theorem 1. CAUTION! Note which ones are the variables of integration! We can denote them as we want, for instance, by τ and p, and write

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \quad \text{and} \quad G(s) = \int_0^\infty e^{-sp} g(p)\, dp.$$

We now set $t = p + \tau$, where τ is at first constant. Then $p = t - \tau$, and t varies from τ to ∞. Thus

$$G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t - \tau)\, dt = e^{s\tau}\int_\tau^\infty e^{-st} g(t - \tau)\, dt.$$

τ in F and t in G vary independently. Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t - \tau)\, dt\, d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t - \tau)\, dt\, d\tau.$$

Here we integrate for fixed τ over t from τ to ∞ and then over τ from 0 to ∞. This is the blue region in Fig. 139. Under the assumption on f and g the order of integration can be reversed (see Ref. [A5] for a proof using uniform convergence). We then integrate first over τ from 0 to t and then over t from 0 to ∞, that is,

$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau) g(t - \tau)\, d\tau\, dt = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}(h) = H(s).$$

This completes the proof. ■

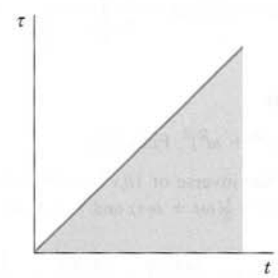


Fig. 139. Region of integration in the $t\tau$-plane in the proof of Theorem 1

From the definition it follows almost immediately that convolution has the properties

$$f * g = g * f \qquad \text{(commutative law)}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)}$$

$$(f * g) * v = f * (g * v) \qquad \text{(associative law)}$$

$$f * 0 = 0 * f = 0$$

similar to those of the multiplication of numbers. Unusual are the following two properties.

EXAMPLE 3 Unusual Properties of Convolution

$f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1\, d\tau = \tfrac{1}{2}t^2 \neq t.$$

$(f * f)(t) \geqq 0$ may not hold. For instance, Example 2 with $\omega = 1$ gives

$$\sin t * \sin t = -\tfrac{1}{2} t \cos t + \tfrac{1}{2} \sin t \qquad \text{(Fig. 140).}$$ ■

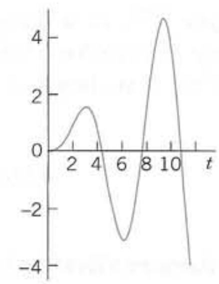


Fig. 140. Example 3

We shall now take up the case of a complex double root (left aside in the last section in connection with partial fractions) and find the solution (the inverse transform) directly by convolution.

EXAMPLE 4 Repeated Complex Factors. Resonance

In an undamped mass–spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is (see Sec. 2.8)

$$y'' + \omega_0^{\,2} y = K \sin \omega_0 t$$

where $\omega_0^{\,2} = k/m$, k is the spring constant, and m is the mass of the body attached to the spring. We assume $y(0) = 0$ and $y'(0) = 0$, for simplicity. Then the subsidiary equation is

$$s^2 Y + \omega_0^{\,2} Y = \frac{K\omega_0}{s^2 + \omega_0^{\,2}}. \qquad \text{Its solution is} \qquad Y = \frac{K\omega_0}{(s^2 + \omega_0^{\,2})^2}.$$

This is a transform as in Example 2 with $\omega = \omega_0$ and multiplied by $K\omega_0$. Hence from Example 2 we can see directly that the solution of our problem is

$$y(t) = \frac{K\omega_0}{2\omega_0^{\,2}} \left(-t \cos \omega_0 t + \frac{\sin \omega_0 t}{\omega_0} \right) = \frac{K}{2\omega_0^{\,2}} \left(-\omega_0 t \cos \omega_0 t + \sin \omega_0 t \right).$$

We see that the first term grows without bound. Clearly, in the case of resonance such a term must occur. (See also a similar kind of solution in Fig. 54 in Sec. 2.8.) ■

Application to Nonhomogeneous Linear ODEs

Nonhomogeneous linear ODEs can now be solved by a general method based on convolution by which the solution is obtained in the form of an integral. To see this, recall from Sec. 6.2 that the subsidiary equation of the ODE

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{2}$$

has the solution [(7) in Sec. 6.2]

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Inversion of the first term $[\cdots]$ provides no difficulty; depending on whether $\frac{1}{4}a^2 - b$ is positive, zero, or negative, its inverse will be a linear combination of two exponential functions, or of the form $(c_1 + c_2 t)e^{-at/2}$, or a damped oscillation, respectively. The interesting term is $R(s)Q(s)$ because $r(t)$ can have various forms of practical importance, as we shall see. If $y(0) = 0$ and $y'(0) = 0$, then $Y = RQ$, and the convolution theorem gives the solution

$$y(t) = \int_0^t q(t - \tau) r(\tau)\, d\tau. \tag{3}$$

EXAMPLE 5 **Response of a Damped Vibrating System to a Single Square Wave**

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) ***that acts for some time only*** (Fig. 141) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t - \tau) \cdot 1\, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \tfrac{1}{2} e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right).$$

Figure 141 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why?), and finally decreases to zero in a monotone fashion. ■

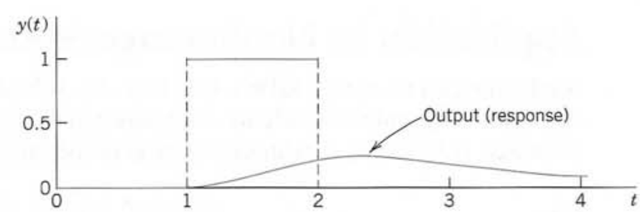


Fig. 141. Square wave and response in Example 5

Integral Equations

Convolution also helps in solving certain **integral equations,** that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it). This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

EXAMPLE 6 A Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau) \sin (t - \tau)\, d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\,\frac{1}{s^2+1} = Y(s)\,\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \quad \text{and gives the answer} \quad y(t) = t + \frac{t^3}{6}.$$

Check the result by a CAS or by substitution and repeated integration by parts (which will need patience). ■

EXAMPLE 7 Another Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation

$$y(t) - \int_0^t (1 + \tau)\, y(t - \tau)\, d\tau = 1 - \sinh t.$$

Solution. By (1) we can write $y - (1 + t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}, \quad \text{hence} \quad Y(s) \cdot \frac{s^2 - s - 1}{s^2} = \frac{s^2 - 1 - s}{s(s^2-1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2-1} \quad \text{and the solution is} \quad y(t) = \cosh t.$$ ■

PROBLEM SET 6.5

1–8 CONVOLUTIONS BY INTEGRATION

Find by integration:

1. $1 * 1$

2. $t * t$

3. $t * e^t$

4. $e^{at} * e^{bt}\ (a \neq b)$

5. $1 * \cos \omega t$

6. $1 * f(t)$

7. $e^{kt} * e^{-kt}$

8. $\sin t * \cos t$

9–16 INVERSE TRANSFORMS BY CONVOLUTION

Find $f(t)$ if $\mathcal{L}(f)$ equals:

9. $\dfrac{1}{(s-3)(s+5)}$

10. $\dfrac{1}{s(s-1)}$

11. $\dfrac{1}{s(s^2+4)}$

12. $\dfrac{1}{s^2(s-2)}$

13. $\dfrac{1}{s^2(s^2+1)}$

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{1}{s(s^2-9)}$

16. $\dfrac{5}{(s^2+1)(s^2+25)}$

17. (Partial fractions) Solve Probs. 9, 11, and 13 by using partial fractions. Comment on the amount of work.

18–25 SOLVING INITIAL VALUE PROBLEMS

Using the convolution theorem, solve:

18. $y'' + y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + 4y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0$

20. $y'' + 5y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0$

[3]If the upper limit of integration is *variable,* the equation is named after the Italian mathematician VITO VOLTERRA (1860–1940), and if that limit is *constant,* the equation is named after the Swedish mathematician IVAR FREDHOLM (1866–1927). "Of the second kind (first kind)" indicates that y occurs (does not occur) outside of the integral.

21. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, \quad y'(0) = 4$

22. $y'' + 3y' + 2y = 1$ if $0 < t < a$ and 0 if $t > a$; $y(0) = 0, \quad y'(0) = 0$

23. $y'' + 4y = 5u(t - 1); \qquad y(0) = 0, \quad y'(0) = 0$

24. $y'' + 5y' + 6y = \delta(t - 3); \quad y(0) = 1, \quad y'(0) = 0$

25. $y'' + 6y' + 8y = 2\delta(t - 1) + 2\delta(t - 2); \quad y(0) = 1, \quad y'(0) = 0$

26. **TEAM PROJECT. Properties of Convolution.** Prove:

(a) Commutativity, $f * g = g * f$

(b) Associativity, $(f * g) * v = f * (g * v)$

(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$

(d) Dirac's delta. Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.

(e) Unspecified driving force. Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \qquad y(0) = K_1, \qquad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t.$$

27–34 INTEGRAL EQUATIONS

Using Laplace transforms and showing the details, solve:

27. $y(t) - \int_0^t y(\tau)\, d\tau = 1$

28. $y(t) + \int_0^t y(\tau) \cosh (t - \tau)\, d\tau = t + e^t$

29. $y(t) - \int_0^t y(\tau) \sin (t - \tau)\, d\tau = \cos t$

30. $y(t) + 2 \int_0^t y(\tau) \cos (t - \tau)\, d\tau = \cos t$

31. $y(t) + \int_0^t (t - \tau) y(\tau)\, d\tau = 1$

32. $y(t) - \int_0^t y(\tau)(t - \tau)\, d\tau = 2 - \frac{1}{2}t^2$

33. $y(t) + 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau = te^t$

34. $y(t) + \int_0^t e^{2(t-\tau)} y(\tau)\, d\tau = t^2 - t - \frac{1}{2} + \frac{1}{2}e^{2t}$

35. **CAS EXPERIMENT. Variation of a Parameter.**
(a) Replace 2 in Prob. 33 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.

(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

The variety of methods for obtaining transforms and inverse transforms and their application in solving ODEs is surprisingly large. We have seen that they include direct integration, the use of linearity (Sec. 6.1), shifting (Secs. 6.1, 6.3), convolution (Sec. 6.5), and differentiation and integration of functions $f(t)$ (Sec. 6.2). But this is not all. In this section we shall consider operations of somewhat lesser importance, namely, differentiation and integration of *transforms* $F(s)$ and corresponding operations for functions $f(t)$, with applications to ODEs with variable coefficients.

Differentiation of Transforms

It can be shown that if a function $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1, then the derivative $F'(s) = dF/ds$ of the transform $F(s) = \mathcal{L}(f)$ can be obtained by differentiating $F(s)$ under the integral sign with respect to s (proof in Ref. [GR4] listed in App. 1). Thus, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad \text{then} \qquad F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

Consequently, if $\mathcal{L}(f) = F(s)$, then

$$\text{(1)} \qquad \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{hence} \qquad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

where the second formula is obtained by applying $\mathcal{L}^{-1}$ on both sides of the first formula. In this way, *differentiation of the transform of a function corresponds to the multiplication of the function by* $-t$.

EXAMPLE 1 Differentiation of Transforms. Formulas 21–23 in Sec. 6.9

We shall derive the following three formulas.

	$\mathcal{L}(f)$	$f(t)$
(2)	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
(3)	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
(4)	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

Solution. From (1) and formula 8 (with $\omega = \beta$) in Table 6.1 of Sec. 6.1 we obtain by differentiation (CAUTION! Chain rule!)

$$\mathcal{L}(t\sin\beta t) = \frac{2\beta s}{(s^2+\beta^2)^2}.$$

Dividing by 2β and using the linearity of $\mathcal{L}$, we obtain (3).

Formulas (2) and (4) are obtained as follows. From (1) and formula 7 (with $\omega = \beta$) in Table 6.1 we find

$$\text{(5)} \qquad \mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2) - 2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}.$$

From this and formula 8 (with $\omega = \beta$) in Table 6.1 we have

$$\mathcal{L}\left(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} \pm \frac{1}{s^2+\beta^2}.$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2 - \beta^2 + s^2 + \beta^2 = 2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2 - \beta^2 - s^2 - \beta^2 = -2\beta^2$, and we obtain (2). This agrees with Example 2 in Sec. 6.5. ■

Integration of Transforms

Similarly, if $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1 and the limit of $f(t)/t$, as t approaches 0 from the right, exists, then for $s > k$,

$$\text{(6)} \qquad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s} \qquad \text{hence} \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}.$$

In this way, *integration of the transform of a function* $f(t)$ *corresponds to the division of* $f(t)$ *by* t.

We indicate how (6) is obtained. From the definition it follows that

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\, dt \right] d\tilde{s},$$

and it can be shown (see Ref. [GR4] in App. 1) that under the above assumptions we may reverse the order of integration, that is,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t)\, d\tilde{s} \right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t}\, d\tilde{s} \right] dt.$$

Integration of $e^{-\tilde{s}t}$ with respect to $\tilde{s}$ gives $e^{-\tilde{s}t}/(-t)$. Here the integral over $\tilde{s}$ on the right equals e^{-st}/t. Therefore,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = \mathcal{L}\left\{ \frac{f(t)}{t} \right\} \qquad (s > k). \quad ■$$

EXAMPLE 2 Differentiation and Integration of Transforms

Find the inverse transform of $\ln\left(1 + \dfrac{\omega^2}{s^2}\right) = \ln \dfrac{s^2 + \omega^2}{s^2}$.

Solution. Denote the given transform by $F(s)$. Its derivative is

$$F'(s) = \frac{d}{ds}\left(\ln(s^2 + \omega^2) - \ln s^2\right) = \frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}.$$

Taking the inverse transform and using (1), we obtain

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{ \frac{2s}{s^2 + \omega^2} - \frac{2}{s} \right\} = 2\cos\omega t - 2 = -tf(t).$$

Hence the inverse $f(t)$ of $F(s)$ is $f(t) = 2(1 - \cos\omega t)/t$. This agrees with formula 42 in Sec. 6.9.

Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}, \qquad \text{then} \qquad g(t) = \mathcal{L}^{-1}(G) = 2(\cos\omega t - 1).$$

From this and (6) we get, in agreement with the answer just obtained,

$$\ln \frac{s^2 + \omega^2}{s^2} = \int_s^\infty G(s)\, ds = -\frac{g(t)}{t} = \frac{2}{t}(1 - \cos\omega t),$$

the minus occurring since s is the lower limit of integration.

In a similar way we obtain formula 43 in Sec. 6.9,

$$\mathcal{L}^{-1}\left\{ \ln\left(1 - \frac{a^2}{s^2}\right) \right\} = \frac{2}{t}(1 - \cosh at). \quad ■$$

Special Linear ODEs with Variable Coefficients

Formula (1) can be used to solve certain ODEs with variable coefficients. The idea is this. Let $\mathcal{L}(y) = Y$. Then $\mathcal{L}(y') = sY - y(0)$ (see Sec. 6.2). Hence by (1),

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}. \tag{7}$$

Similarly, $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$ and by (1)

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0). \tag{8}$$

Hence if an ODE has coefficients such as $at + b$, the subsidiary equation is a first-order ODE for Y, which is sometimes simpler than the given second-order ODE. But if the latter has coefficients $at^2 + bt + c$, then two applications of (1) would give a second-order ODE for Y, and this shows that the present method works well only for rather special ODEs with variable coefficients. An important ODE for which the method is advantageous is the following.

EXAMPLE 3 Laguerre's Equation. Laguerre Polynomials

Laguerre's ODE is

$$ty'' + (1 - t)y' + ny = 0. \tag{9}$$

We determine a solution of (9) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get the subsidiary equation

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken zero), and taking exponentials, we get

$$\frac{dY}{Y} = -\frac{n + 1 - s}{s - s^2}\,ds = \left(\frac{n}{s - 1} - \frac{n + 1}{s}\right)ds \quad \text{and} \quad Y = \frac{(s - 1)^n}{s^{n+1}}. \tag{10*}$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots. \tag{10}$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 5.7, but we continue to reserve capital letters for transforms). We prove (10). By Table 6.1 and the first shifting theorem (s-shifting),

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s + 1)^{n+1}}, \qquad \text{hence by (3) in Sec. 6.2} \qquad \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!\,s^n}{(s + 1)^{n+1}}$$

because the derivatives up to the order $n - 1$ are zero at 0. Now make another shift and divide by $n!$ to get [see (10) and then (10*)]

$$\mathcal{L}(l_n) = \frac{(s - 1)^n}{s^{n+1}} = Y. \qquad \blacksquare$$

PROBLEM SET 6.6

1–12 TRANSFORMS BY DIFFERENTIATION

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

1. $4te^t$
2. $-t \cosh 2t$
3. $t \sin \omega t$
4. $t \cos (t + k)$
5. $te^{-2t} \sin t$
6. $t^2 \sin 3t$
7. $t^2 \sinh 4t$
8. $t^n e^{kt}$
9. $t^2 \sin \omega t$
10. $t \cos \omega t$
11. $t \sin (t + k)$
12. $te^{-kt} \sin t$

13–20 INVERSE TRANSFORMS

Using differentiation, integration, s-shifting, or convolution (and showing the details), find $f(t)$ if $\mathcal{L}(f)$ equals:

13. $\dfrac{6}{(s+1)^2}$

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{2(s+2)}{[(s+2)^2+1]^2}$

16. $\dfrac{s}{(s^2-1)^2}$

17. $\dfrac{2}{(s-k)^3}$

18. $\ln \dfrac{s+a}{s+b}$

19. $\ln \dfrac{s}{s-1}$

20. $\operatorname{arccot} \dfrac{s}{\omega}$

21. WRITING PROJECT. Differentiation and Integration of Functions and Transforms. Make a short draft of these four operations from memory. Then compare your notes with the text and write a report of 2–3 pages on these operations and their significance in applications.

22. CAS PROJECT. Laguerre Polynomials. (a) Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

(b) Show that

$$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} t^m$$

and calculate $l_0, \cdots, l_{10}$ from this formula.

(c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

$$(n+1)l_{n+1} = (2n+1-t)l_n - nl_{n-1}.$$

(d) Experiment with the graphs of $l_0, \cdots, l_{10}$, finding out empirically how the first maximum, first minimum, $\cdots$ is moving with respect to its location as a function of n. Write a short report on this.

(e) A **generating function** (definition in Problem Set 5.3) for the Laguerre polynomials is

$$\sum_{n=0}^{\infty} l_n(t)x^n = (1-x)^{-1}e^{tx/(x-1)}.$$

Obtain $l_0, \cdots, l_{10}$ from the corresponding partial sum of this power series in x and compare the l_n with those in (a), (b), or (c).

6.7 Systems of ODEs

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned} \tag{1}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\begin{aligned} (a_{11}-s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22}-s)Y_2 &= -y_2(0) - G_2(s). \end{aligned} \tag{2}$$

By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).

Note that (1) and (2) may be written in vector form (and similarly for the systems in the examples); thus, setting $\mathbf{y} = [y_1 \quad y_2]^\mathsf{T}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \quad g_2]^\mathsf{T}$, $\mathbf{Y} = [Y_1 \quad Y_2]^\mathsf{T}$, $\mathbf{G} = [G_1 \quad G_2]^\mathsf{T}$ we have

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad \text{and} \qquad (\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}.$$

EXAMPLE 1 **Mixing Problem Involving Two Tanks**

Tank T_1 in Fig. 142 contains initially 100 gal of pure water. Tank T_2 contains initially 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Solution. The model is obtained in the form of two equations

$$\text{Time rate of change} = \text{Inflow/min} - \text{Outflow/min}$$

for the two tanks (see Sec. 4.1). Thus,

$$y_1' = -\frac{8}{100}\,y_1 + \frac{2}{100}\,y_2 + 6, \qquad y_2' = \frac{8}{100}\,y_1 - \frac{8}{100}\,y_2.$$

The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this we see that the subsidiary system (2) is

$$\begin{aligned} (-0.08 - s)Y_1 + \quad 0.02Y_2 \quad &= -\frac{6}{s} \\ 0.08Y_1 \quad + (-0.08 - s)Y_2 &= -150. \end{aligned}$$

We solve this algebraically for Y_1 and Y_2 by elimination (or by Cramer's rule in Sec. 7.7), and we write the solutions in terms of partial fractions,

$$Y_1 = \frac{9s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}.$$

By taking the inverse transform we arrive at the solution

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

Figure 142 shows the interesting plot of these functions. Can you give physical explanations for their main features? Why do they have the limit 100? Why is y_2 not monotone, whereas y_1 is? Why is y_1 from some time on suddenly larger than y_2? Etc. ■

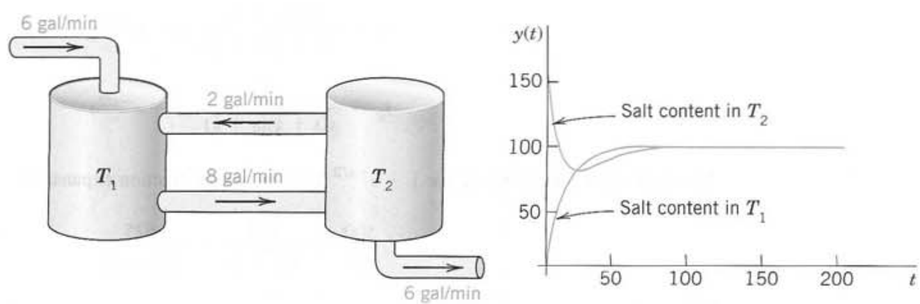


Fig. 142. Mixing problem in Example 1

Other systems of ODEs of practical importance can be solved by the Laplace transform method in a similar way, and eigenvalues and eigenvectors as we had to determine them in Chap. 4 will come out automatically, as we have seen in Example 1.

EXAMPLE 2 Electrical Network

Find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig. 143 with L and R measured in terms of the usual units (see Sec. 2.9), $v(t) = 100$ volts if $0 \leqq t \leqq 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.

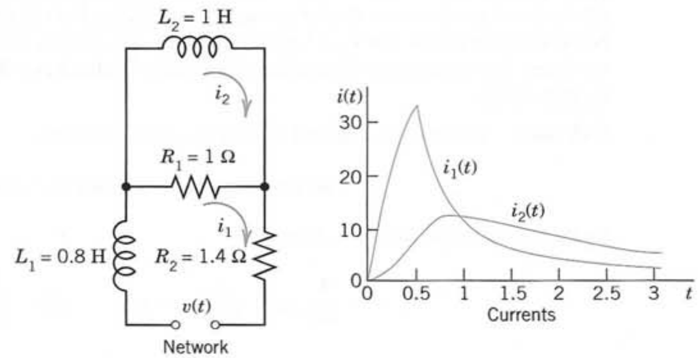


Fig. 143. Electrical network in Example 2

Solution. The model of the network is obtained from Kirchhoff's voltage law as in Sec. 2.9. For the lower circuit we obtain

$$0.8\, i_1' + 1(i_1 - i_2) + 1.4\, i_1 = 100[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$1 \cdot i_2' + 1(i_2 - i_1) \qquad = 0.$$

Division by 0.8 and ordering gives for the lower circuit

$$i_1' + 3i_1 - 1.25\, i_2 = 125[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$i_2' - i_1 + \quad i_2 = 0.$$

With $i_1(0) = 0$, $i_2(0) = 0$ we obtain from (1) in Sec. 6.2 and the second shifting theorem the subsidiary system

$$\begin{aligned} (s + 3) I_1 - \ 1.25 I_2 &= 125 \left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right) \\ -I_1 + (s + 1) I_2 &= 0. \end{aligned}$$

Solving algebraically for I_1 and I_2 gives

$$I_1 = \frac{125(s + 1)}{s(s + \frac{1}{2})(s + \frac{7}{2})} (1 - e^{-s/2}),$$

$$I_2 = \frac{125}{s(s + \frac{1}{2})(s + \frac{7}{2})} (1 - e^{-s/2}).$$

The right sides without the factor $1 - e^{-s/2}$ have the partial fraction expansions

$$\frac{500}{7s} - \frac{125}{3(s + \frac{1}{2})} - \frac{625}{21(s + \frac{7}{2})}$$

and

$$\frac{500}{7s} - \frac{250}{3(s+\frac{1}{2})} + \frac{250}{21(s+\frac{7}{2})},$$

respectively. The inverse transform of this gives the solution for $0 \leqq t \leqq \frac{1}{2}$,

$$\begin{aligned} i_1(t) &= -\frac{125}{3}e^{-t/2} - \frac{625}{21}e^{-7t/2} + \frac{500}{7} \\ i_2(t) &= -\frac{250}{3}e^{-t/2} + \frac{250}{21}e^{-7t/2} + \frac{500}{7} \end{aligned} \qquad (0 \leqq t \leqq \tfrac{1}{2}).$$

According to the second shifting theorem the solution for $t > \frac{1}{2}$ is $i_1(t) - i_1(t - \frac{1}{2})$ and $i_2(t) - i_2(t - \frac{1}{2})$, that is,

$$\begin{aligned} i_1(t) &= -\frac{125}{3}(1 - e^{1/4})e^{-t/2} - \frac{625}{21}(1 - e^{7/4})e^{-7t/2} \\ i_2(t) &= -\frac{250}{3}(1 - e^{1/4})e^{-t/2} + \frac{250}{21}(1 - e^{7/4})e^{-7t/2} \end{aligned} \qquad (t > \tfrac{1}{2}).$$

Can you explain physically why both currents eventually go to zero, and why $i_1(t)$ has a sharp cusp whereas $i_2(t)$ has a continuous tangent direction at $t = \frac{1}{2}$? ■

Systems of ODEs of higher order can be solved by the Laplace transform method in a similar fashion. As an important application, typical of many similar mechanical systems, we consider coupled vibrating masses on springs.

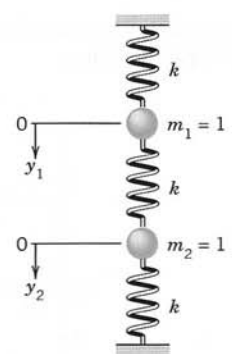


Fig. 144. Example 3

EXAMPLE 3 Model of Two Masses on Springs (Fig. 144)

The mechanical system in Fig. 144 consists of two bodies of mass 1 on three springs of the same spring constant k and of negligibly small masses of the springs. Also damping is assumed to be practically zero. Then the model of the physical system is the system of ODEs

$$\begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2. \end{aligned} \tag{3}$$

Here y_1 and y_2 are the displacements of the bodies from their positions of static equilibrium. These ODEs follow from **Newton's second law,** *Mass × Acceleration = Force,* as in Sec. 2.4 for a single body. We again regard downward forces as positive and upward as negative. On the upper body, $-ky_1$ is the force of the upper spring and $k(y_2 - y_1)$ that of the middle spring, $y_2 - y_1$ being the net change in spring length—think this over before going on. On the lower body, $-k(y_2 - y_1)$ is the force of the middle spring and $-ky_2$ that of the lower spring.

We shall determine the solution corresponding to the initial conditions $y_1(0) = 1$, $y_2(0) = 1$, $y_1'(0) = \sqrt{3k}$, $y_2'(0) = -\sqrt{3k}$. Let $Y_1 = \mathcal{L}(y_1)$ and $Y_2 = \mathcal{L}(y_2)$. Then from (2) in Sec. 6.2 and the initial conditions we obtain the subsidiary system

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2.$$

This system of linear algebraic equations in the unknowns Y_1 and Y_2 may be written

$$(s^2 + 2k)Y_1 - kY_2 = s + \sqrt{3k}$$
$$-kY_1 + (s^2 + 2k)Y_2 = s - \sqrt{3k}.$$

Elimination (or Cramer's rule in Sec. 7.7) yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is (Fig. 145)

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$
$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation. ■

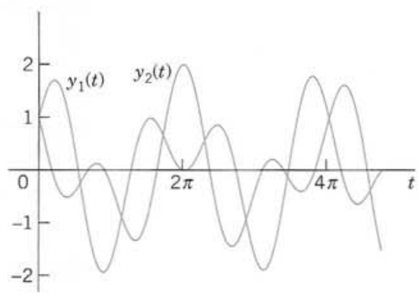


Fig. 145. Solutions in Example 3

PROBLEM SET 6.7

1–20 SYSTEMS OF ODES

Using the Laplace transform and showing the details of your work, solve the initial value problem:

1. $y_1' = -y_1 - y_2$, $y_2' = y_1 - y_2$, $y_1(0) = 0$, $y_2(0) = 1$
2. $y_1' = 5y_1 + y_2$, $y_2' = y_1 + 5y_2$, $y_1(0) = 1$, $y_2(0) = -3$
3. $y_1' = -6y_1 + 4y_2$, $y_2' = -4y_1 + 4y_2$, $y_1(0) = -2$, $y_2(0) = -7$
4. $y_1' + y_2 = 0$, $y_1 + y_2' = 2\cos t$, $y_1(0) = 1$, $y_2(0) = 0$
5. $y_1' = -4y_1 - 2y_2 + t$, $y_2' = 3y_1 + y_2 - t$, $y_1(0) = 5.75$, $y_2(0) = -6.75$
6. $y_1' = 4y_2 - 8\cos 4t$, $y_2' = -3y_1 - 9\sin 4t$, $y_1(0) = 0$, $y_2(0) = 3$

7. $y_1' = 5y_1 - 4y_2 - 9t^2 + 2t,$
$y_2' = 10y_1 - 7y_2 - 17t^2 - 2t,$
$y_1(0) = 2, \quad y_2(0) = 0$

8. $y_1' = 6y_1 + y_2, \quad y_2' = 9y_1 + 6y_2,$
$y_1(0) = -3, \quad y_2(0) = -3$

9. $y_1' = 5y_1 + 5y_2 - 15 \cos t + 27 \sin t,$
$y_2' = -10y_1 - 5y_2 - 150 \sin t,$
$y_1(0) = 2, \quad y_2(0) = 2$

10. $y_1' = -2y_1 + 3y_2, \quad y_2' = 4y_1 - y_2,$
$y_1(0) = 4, \quad y_2(0) = 3$

11. $y_1' = y_2 + 1 - u(t - 1),$
$y_2' = -y_1 + 1 - u(t - 1), \quad y_1(0) = 0,$
$y_2(0) = 0$

12. $y_1' = 2y_1 + y_2, \quad y_2' = 4y_1 + 2y_2 + 64tu(t - 1),$
$y_1(0) = 2, \quad y_2(0) = 0$

13. $y_1' = y_1 + 6u(t - 2)e^{4t}, \quad y_2' = y_1 + 2y_2,$
$y_1(0) = 0, \quad y_2(0) = 1$

14. $y_1' = -y_2, \quad y_2' = -y_1 + 2[1 - u(t - 2\pi)] \cos t,$
$y_1(0) = 1, \quad y_2(0) = 0$

15. $y_1' = -3y_1 + y_2 + u(t - 1)e^t,$
$y_2' = -4y_1 + 2y_2 + u(t - 1)e^t,$
$y_1(0) = 0, \quad y_2(0) = 3$

16. $y_1'' = -2y_1 + 2y_2, \quad y_2'' = 2y_1 - 5y_2,$
$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 3, \quad y_2'(0) = 0$

17. $y_1'' = 4y_1 + 8y_2, \quad y_2'' = 5y_1 + y_2,$
$y_1(0) = 8, \quad y_1'(0) = -18, \quad y_2(0) = 5,$
$y_2'(0) = -21$

18. $y_1'' + y_2 = -101 \sin 10t, \quad y_2'' + y_1 = 101 \sin 10t,$
$y_1(0) = 0, \quad y_1'(0) = 6, \quad y_2(0) = 8,$
$y_2'(0) = -6$

19. $y_1' + y_2' = 2e^t + e^{-t}, \quad y_2' + y_3' = 2 \sinh t,$
$y_3' + y_1' = e^t$
$y_1(0) = 0, \quad y_2(0) = 1, \quad y_3(0) = 1$

20. $4y_1' + y_2' - 2y_3' = 0, \quad -2y_1' + y_3' = 1,$
$2y_2' - 4y_3' = -16t$
$y_1(0) = 2, \quad y_2(0) = 0, \quad y_3(0) = 0$

21. TEAM PROJECT. Comparison of Methods for Linear Systems of ODEs.

(a) Models. Solve the models in Examples 1 and 2 of Sec. 4.1 by Laplace transforms and compare the amount of work with that in Sec. 4.1. (Show the details of your work.)

(b) Homogeneous Systems. Solve the systems (8), (11)–(13) in Sec. 4.3 by Laplace transforms. (Show the details.)

(c) Nonhomogeneous System. Solve the system (3) in Sec. 4.6 by Laplace transforms. (Show the details.)

FURTHER APPLICATIONS

22. (Forced vibrations of two masses) Solve the model in Example 3 with $k = 4$ and initial conditions $y_1(0) = 1$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = -1$ under the assumption that the force $11 \sin t$ is acting on the first body and the force $-11 \sin t$ on the second. Graph the two curves on common axes and explain the motion physically.

23. CAS Experiment. Effect of Initial Conditions. In Prob. 22, vary the initial conditions systematically, describe and explain the graphs physically. The great variety of curves will surprise you. Are they always periodic? Can you find empirical laws for the changes in terms of continuous changes of those conditions?

24. (Mixing problem) What will happen in Example 1 if you double all flows (in particular, an increase to 12 gal/min containing 12 lb of salt from the outside), leaving the size of the tanks and the initial conditions as before? First guess, then calculate. Can you relate the new solution to the old one?

25. (Electrical network) Using Laplace transforms, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 146, where $v(t) = 390 \cos t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state?

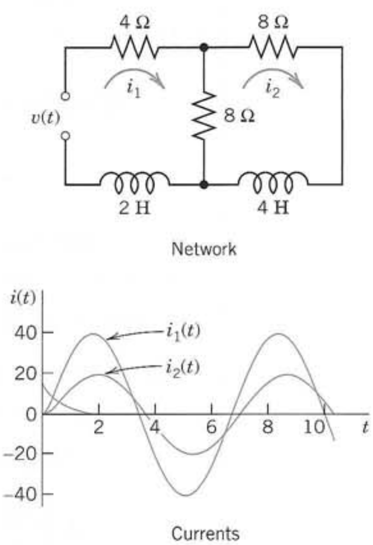


Fig. 146. Electrical network and currents in Problem 25

26. (Single cosine wave) Solve Prob. 25 when the EMF (electromotive force) is acting from 0 to 2π only. Can you do this just by looking at Prob. 25, practically without calculation?

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$ $= \int_0^t f(t - \tau)g(\tau)\, d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\, u(t - a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref.* [A9] *in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	6.1
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	6.1
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	6.1
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	6.1
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	6.1
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	6.1
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1}e^{at}$	6.1
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$	6.1
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)} (e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)} (ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2 + \omega^2}$	$\dfrac{1}{\omega} \sin \omega t$	6.1
14	$\dfrac{s}{s^2 + \omega^2}$	$\cos \omega t$	6.1
15	$\dfrac{1}{s^2 - a^2}$	$\dfrac{1}{a} \sinh at$	6.1
16	$\dfrac{s}{s^2 - a^2}$	$\cosh at$	6.1
17	$\dfrac{1}{(s-a)^2 + \omega^2}$	$\dfrac{1}{\omega} e^{at} \sin \omega t$	6.1
18	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	6.1
19	$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2} (1 - \cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2 + \omega^2)}$	$\dfrac{1}{\omega^3} (\omega t - \sin \omega t)$	6.2
21	$\dfrac{1}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$	6.6

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin\omega t$	6.6
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin\omega t+\omega t\cos\omega t)$	6.6
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}\quad(a^2\neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at-\cos bt)$	
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt\cosh kt-\cos kt\sinh kt)$	
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2}\sin kt\sinh kt$	
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt-\sin kt)$	
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt-\cos kt)$	
29	$\sqrt{s-a}-\sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2}I_0\left(\dfrac{a-b}{2}t\right)$	5.6
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	5.5
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k}\quad(k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2}I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}}\quad(k>0)$	$\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$	
40	$\dfrac{1}{s}\ln s$	$-\ln t-\gamma\quad(\gamma\approx 0.5772)$	5.6

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$	
45	$\frac{1}{s}\,\text{arccot}\, s$	$\text{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. What do we mean by operational calculus?
2. What are the steps needed in solving an ODE by Laplace transform? What is the subsidiary equation?
3. The Laplace transform is a linear operation. What does this mean? Why is it important?
4. For what problems is the Laplace transform preferable over the usual method? Explain.
5. What are the unit step and Dirac's delta functions? Give examples.
6. What is the difference between the two shifting theorems? When do they apply?
7. Is $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
8. Can a discontinuous function have a Laplace transform? Does every continuous function have a Laplace transform? Give reasons.
9. State the transforms of a few simple functions from memory.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–22 LAPLACE TRANSFORMS

Find the transform (showing the details of your work and indicating the method or formula you are using):

11. te^{3t}
12. $e^{-t} \sin 2t$
13. $\sin^2 t$
14. $\cos^2 4t$
15. $tu(t - \pi)$
16. $u(t - 2\pi) \sin t$
17. $e^t * \cos 2t$
18. $(\sin \omega t) * (\cos \omega t)$
19. $\sin t + \sinh t$
20. $\cosh t - \cos t$
21. $e^{at} - e^{bt} \quad (a \neq b)$
22. $\cosh 2t - \cosh t$

23–34 INVERSE LAPLACE TRANSFORMS

Find the inverse transform (showing the details of your work and indicating the method or formula used):

23. $\dfrac{10s}{s^2+2}$
24. $\dfrac{15}{s^2-4}$
25. $\dfrac{12}{s^2+4s+20}$
26. $\dfrac{3s}{s^2-2s+2}$
27. $\dfrac{5s+4}{s^2} e^{-2s}$
28. $\dfrac{2s-10}{s^3} e^{-5s}$
29. $\dfrac{2s+4}{(s^2+4s+5)^2}$
30. $\dfrac{s^2-16}{(s^2+16)^2}$
31. $\left(\dfrac{2}{s^2} + \dfrac{2}{s^3}\right) e^{-s}$
32. $\dfrac{180 + 18s^2 + 3s^4}{s^7}$
33. $\dfrac{\pi}{s^2(s^2+\omega^2)}$
34. $\dfrac{2}{2s^2+2s+1}$

35–50 **SINGLE ODEs AND SYSTEMS OF ODEs**

Solve by Laplace transforms, showing the details and graphing the solution:

35. $y'' + y = u(t-1)$, $y(0) = 0$, $y'(0) = 20$

36. $y'' + 16y = 4\delta(t-\pi)$, $y(0) = -1$, $y'(0) = 0$

37. $y'' + 4y = 8\delta(t-5)$, $y(0) = 10$, $y'(0) = -1$

38. $y'' + y = u(t-2)$, $y(0) = 0$, $y'(0) = 0$

39. $y'' + 2y' + 10y = 0$, $y(0) = 7$, $y'(0) = -1$

40. $y'' + 4y' + 5y = 50t$, $y(0) = 5$, $y'(0) = -5$

41. $y'' - y' - 2y = 12u(t-\pi)\sin t$, $y(0) = 1$, $y'(0) = -1$

42. $y'' - 2y' + y = t\delta(t-1)$, $y(0) = 0$, $y'(0) = 0$

43. $y'' - 4y' + 4y = \delta(t-1) - \delta(t-2)$, $y(0) = 0$, $y'(0) = 0$

44. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$

45. $y_1' + y_2 = \sin t$, $y_2' + y_1 = -\sin t$, $y_1(0) = 1$, $y_2(0) = 0$

46. $y_1' = -3y_1 + y_2 - 12t$, $y_2' = -4y_1 + 2y_2 + 12t$, $y_1(0) = 0$, $y_2(0) = 0$

47. $y_1' = y_2$, $y_2' = -5y_1 - 2y_2$, $y_1(0) = 0$, $y_2(0) = 1$

48. $y_1' = y_2$, $y_2' = -4y_1 + \delta(t-\pi)$, $y_1(0) = 0$, $y_2(0) = 0$

49. $y_1'' = 4y_2 - 4e^t$, $y_2'' = 3y_1 + y_2$, $y_1(0) = 1$, $y_1'(0) = 2$, $y_2(0) = 2$, $y_2'(0) = 3$

50. $y_1'' = 16y_2$, $y_2'' = 16y_1$, $y_1(0) = 2$, $y_1'(0) = 12$, $y_2(0) = 6$, $y_2'(0) = 4$

MODELS OF CIRCUITS AND NETWORKS

51. **(*RC*-circuit)** Find and graph the current $i(t)$ in the *RC*-circuit in Fig. 147, where $R = 100\ \Omega$, $C = 10^{-3}$ F, $v(t) = 100t$ V if $0 < t < 2$, $v(t) = 200$ V if $t > 2$ and the initial charge on the capacitor is 0.

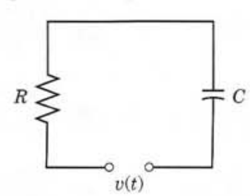


Fig. 147. *RC*-circuit

52. **(*LC*-circuit)** Find and graph the charge $q(t)$ and the current $i(t)$ in the *LC*-circuit in Fig. 148, where $L = 0.5$ H, $C = 0.02$ F, $v(t) = 1425 \sin 5t$ V if $0 < t < \pi$, $v(t) = 0$ if $t > \pi$, and current and charge at $t = 0$ are 0.

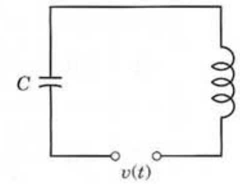


Fig. 148. *LC*-circuit

53. **(*RLC*-circuit)** Find and graph the current $i(t)$ in the *RLC*-circuit in Fig. 149, where $R = 1\ \Omega$, $L = 0.25$ H, $C = 0.2$ F, $v(t) = 377 \sin 20t$ V, and current and charge at $t = 0$ are 0.

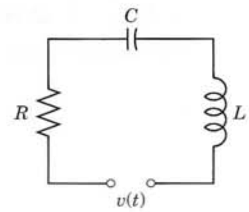


Fig. 149. *RLC*-circuit

54. **(Network)** Show that by Kirchhoff's voltage law (Sec. 2.9), the currents in the network in Fig. 150 are obtained from the system

$$Li_1' + R(i_1 - i_2) = v(t)$$

$$R(i_2' - i_1') + \frac{1}{C} i_2 = 0.$$

Solve this system, where $R = 1\ \Omega$, $L = 2$ H, $C = 0.5$ F, $v(t) = 90e^{-t/4}$ V, $i_1(0) = 0$, $i_2(0) = 2$ A.

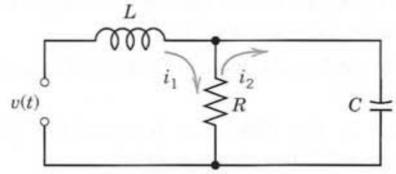


Fig. 150. Network in Problem 54

55. **(Network)** Set up the model of the network in Fig. 151 and find and graph the currents, assuming that the currents and the charge on the capacitor are 0 when the switch is closed at $t = 0$.

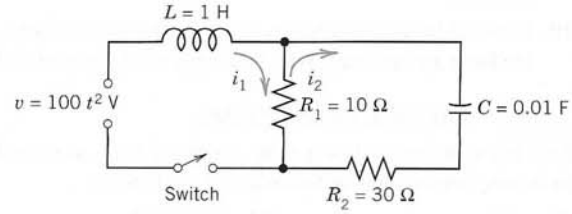


Fig. 151. Network in Problem 55

SUMMARY OF CHAPTER 6

Laplace Transforms

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{(Sec. 6.1).} \tag{1}$$

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\begin{aligned} \mathcal{L}(f') &= s\mathcal{L}(f) - f(0) \\ \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0) \end{aligned} \qquad \text{(Sec. 6.2)} \tag{2}$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad (a,\ b \text{ constant}) \tag{3}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0). \tag{4}$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the small table in Sec. 6.1 or the larger table in Sec. 6.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform** $y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $\mathcal{L}^{-1}(Y)$ by partial fraction reduction (Sec. 6.4) if we see no simpler way.

The Laplace method avoids the determination of a general solution of the homogeneous ODE, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4). Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 6.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Secs. 6.3 and 6.4, and more on convolution in Sec. 6.5. Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leqq t \leqq 4\pi$ and 0 elsewhere.

The application of the Laplace transform to systems of ODEs is shown in Sec. 6.7. (The application to PDEs follows in Sec. 12.11.)