The Journal of Fuzzy Mathematics Vol. 20, No. 1, 2012
Los Angeles

Sets Having Finite Fuzzy Measure in Real Hilbert Spaces

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Abstract:
A new type of translation invariant and lower semi continuous fuzzy measure on the class of subsets of a real Hilbert space is introduced. It measures a subset of the Hilbert space as a projection of the set along a fixed vector in the Hilbert space. It is proved that corresponding to each subset of the Hilbert space, the fuzzy measure is determined by one vector of the Hilbert space. Then it is proved that the fuzzy measure of a closed convex subset of the Hilbert space can be obtained in terms of two elements of the subset itself. It is also proved that this fuzzy measure satisfies a condition similar to the null additivity.

Keywords:
Vector Generated Fuzzy Measure, Null additivity, Support space, Null space, Closed and Convex subsets of a Hilbert space.

1. Introduction

A fuzzy measure can be viewed as a generalization of the measurement like length, area, volume etc. of objects in the three dimensional space. Since the real Hilbert spaces form a generalization of the three dimensional Euclidean space which contains all the real and observable objects, it is natural to study fuzzy measures on a Hilbert space. Manju Cherian and K. Sudheer [3] had introduced the *Vector Generated Fuzzy Measure* $(VGFM)$ on a real Hilbert space and studied some of its properties. For closed convex subsets of a real Hilbert space, the $VGFM$ becomes a generalization of the length of an interval in R as the difference between two points in the set. Throughout in this discussion, H is assumed to be a real Hilbert space.

Received September, 2010

1066-8950/12 $8.50

2. Fuzzy measures

Definition 2.1 (Fuzzy measure) [8]. Let X is a non-empty set and C, a class of subsets of X. A fuzzy measure is an extended real valued function $\mu : C \to [0,\infty]$ satisfying the conditions

FM_1: $\mu(\Phi) = 0$ whenever $\Phi \in C$ and

FM_2 : (**monotonicity**) For $A, B \in C$, $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$.

The lower and upper semi-continuity of a fuzzy measure are respectively defined as follows.

FM_3: (**lower semi-continuity**) If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots \in C$ and if $E = \bigcup_{n=1}^{\infty} E_n \in C$ then $\lim_{n\to\infty} \mu(E_n) = \mu(E)$;

FM_4 : (**upper semi-continuity**) If $E_1, E_2, \cdots, E_n, \cdots \in C$ are such that $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$, $\mu(E_k) < \infty$ for some k and if $E = \bigcap_{n=1}^{\infty} \in C$ then $\lim_{n\to\infty} \mu(E_n) = \mu(E)$.

Note 2.2. A fuzzy measure need not satisfy the additivity property which a measure usually possesses.

Definition 2.3 (Null Additivity). Let $\mathcal{A}$ be a σ-algebra of subsets of a non-empty set X. A function $\mu : \mathcal{A} \to [-\infty,\infty]$ is *null additive* if $\mu(E \cup F) = \mu(E)$ whenever $E \in \mathcal{A}$, $F \in \mathcal{A}$, $E \cap F = \Phi$ and $\mu(F) = 0$

3. Fuzzy measures on Hilbert spaces

Definition 3.1 (Vector generated fuzzy measure $(VGFM)$) [3]. If H is a Hilbert Space and if $x \in H$ is a unit vector, a fuzzy measure μ_x generated by the vector x or a Vector Generated Fuzzy Measure $(VGFM)$ is defined as

$$\mu_x(A) = \sup\{\langle a - b, x\rangle : a, b \in A\}$$

for any $A \subseteq H$.

Remark 3.2. It measures every subset A of the Hilbert space H as a projection along the vector x. A $VGFM$ holds the following properties; [3].

[1] It is a fuzzy measure;

[2] It is defined on the power set of the Hilbert space;

[3] It is translation invariant;

[4] There exists two closed subspaces V and W of H respectively called the *support space* and *null space* of μ_x satisfying

(a) $x \in V$ and $x \perp W$,

(b) $V \perp W$,

(c) $\mu_x(A) \neq 0$ only when $A \subseteq V$ and $\mu_x(B) = 0$ for all $B \subseteq W$. (Take $W = \left[\{y \in H : \langle y, x\rangle = 0\}\right]$ and $V = W^{\perp}$. It is possible that both V and W to be non-zero spaces).

(5) $VGFM$ of a singleton is zero;

(6) It is a lower semi-continuous fuzzy measure;

(7) It is not an additive fuzzy measure;

(8) It is not null additive;

As an example, take $H = R^3$, $E = \{(a,b,0) : 0 \le a \le 1, 0 \le b \le 1\}$, $F = \{(2,2,0)\}$ and $x = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$, then $\mu_x(E) = \sqrt{2}$, $\mu_x(F) = 0$ and $E \cap F = \Phi$. But $\mu_x(E \cup F)$ $\neq \mu_x(E)$. (Here, $\mu_x(E \cup F) = 2\sqrt{2}$).

Theorem 3.3. *Let μ_x be a $VGFM$ defined on a Hilbert space H. Let A be a subset of the support space V of μ_x. For any set $A \subset V$, $\mu_x(A) = 0$ if and only if A is a singleton.*

Proof. If $A \subset V$ is a singleton, from the definition of a $VGFM$, it follows that $\mu_x(A)$ $= 0$. Conversely, assume that $\mu_x(A) = 0$. Then for each pair $a, b \in A$, $\langle a-b, x\rangle \le 0$ since 0 is the supremum. In case $\langle a-b, x\rangle < 0$ for a pair $a, b \in A$, then $\langle b-a, x\rangle > 0$ which contradicts the fact that $\mu_x(A) = 0$. Thus $\langle a-b, x\rangle = 0$ for all $a, b \in A$. Then $a - b \in W$, the null space of μ_x for all $a, b \in A$. But $a, b \in A \subseteq V \Rightarrow a - b \in V$. Since $V \perp W$, it follows that $a - b \in V \cap W$. Thus $a - b = 0$ or $a = b$. Since $a, b \in A$ are arbitrary, it follows that A is a singleton.

Corollary 3.4. *If A is a subset of the Hilbert space H, then $\mu_x(A) = 0$ if and only if there is $v \in V$ with $A = v \oplus W$ where V, W are respectively the support and null spaces of μ_x.*

Proof. The proof is an appropriate blend of the following facts:

- μ_x is a translation invariant fuzzy measure;
- $\mu_x(B) = 0$ for all $B \subseteq W$; and
- For $A \subset V$, $\mu_x(A) = 0$ if and only if A is a singleton.

The next theorem suggests that a $VGFM$ satisfies null additivity in a special form.

Theorem 3.5. *Let $\mathcal{A}$ be a σ-algebra of subsets of a Hilbert space H. For any unit vector $x \in H$, the $VGFM$ μ_x satisfies the condition*

$$\mu_x(E \cup F) = \mu_x(E)$$

whenever $E \,\&\, E \cup F \in \mathcal{A}$ are convex sets, $F \in \mathcal{A}$, $E \cap F = \Phi$ and $\mu_x(F) = 0$.

Proof. Since subsets of W ,the null space of μ_x , have their $VGFM$ zero, it is enough to consider the case when both E & F are subsets of V , the support space of μ_x . It is possible to consider $F=\{f\}$, using the fact that $\mu_x(F)=0 \Rightarrow F$ is a singleton. If $\mu_x(E\cup F)=\mu_x(E)$ then there is noting to prove. Let us assume, on the other hand, that $\mu_x(E\cup F)\neq \mu_x(E)$. Since $E\subset E\cup F$, we have $\mu_x(E\cup F)>\mu_x(E)$ (by the mono-tonicity of fuzzy measure). Let α be a real number satisfying $\mu_x(E\cup F)>\alpha>\mu_x(E)$. Then for all $a,b\in E$, the inequality $\langle a-b,x\rangle<\alpha$ holds and by definition of $\mu_x(E\cup F)$ as a supremum, there must exist $a\in E$ such that $\langle a-f,x\rangle>\alpha$. (Otherwise, the case $\langle f-a,x\rangle>\alpha$ can be considered to prove the existence of $a\in E$). Since $E\cup F$ is a convex set, for any $0\le t\le 1$, the vector $a_t=(1-t)a+tf\in E\cup F$. Clearly $a_1=f$ and using the convexity of E , the vector $a_t\in E$ for $0\le t<1$. Now $\langle a-a_t,x\rangle<\alpha$ for $0\le t<1$. By the continuity of the linear functional $y\mapsto\langle y,x\rangle$, it follows that $\langle a-f,x\rangle=\lim_{t\to 1}\langle a-a_t,x\rangle\le\alpha$. Again since $\lim_{t\to 1}a_t=f$, it follows that $\lim_{t\to 1}\langle a-a_t,x\rangle=\langle a-f,x\rangle>\alpha$, which contradics the condition $\langle a-f,x\rangle\le\alpha$. Thus it is proved that $\mu_x(E\cup F)=\mu_x(E)$.

4. Sets having finite $VGFM$

If μ_x is a $VGFM$ defined on a Hilbert space H , then corresponding to each subset of the Hilbert space having finite measure, there exists a vector $h\in H$, producing the measure of the subset. Moreover, if the subset is closed and convex, there exists one pair of elements of the subset, producing the measure of the subset. This is apparently a generalization of the fact that the length of an interval in R is the difference between its end points.

Theorem 4.1. *Let μ_x be a $VGFM$ defined on H . If $\mu_x(A)<\infty$ for $A\subseteq H$, then there exists a vector $h\in H$ such that $\mu_x(A)=\langle h,x\rangle$.*

Proof. Since x is a unit vector in H , the mapping $y\mapsto\langle y,x\rangle$ is a continuous non-zero linear functional of H onto R . So there exists a vector $h\in H$ satisfying $\mu_x(A)=\langle h,x\rangle$.

Corollary 4.2. I*f h , and h' are two vectors in H such that $\langle h,x\rangle=\mu_x(A)=\langle h',x\rangle$, then $h-h'\in W$, the null space of the $VGFM$ μ_x . In other words, the uniqueness of the vector producing the $VGFM$ of a set of finite measure is upto a translation by elements of the null space of the $VGFM$.*

Proof. $\langle h,x\rangle=\langle h',x\rangle$ implies that $\langle h-h',x\rangle=0$ so that $h-h'\in W$.

Remark 4.3. The vector $h \in H$ satisfying $\mu_x(A) = \langle h, x \rangle$ need not be unique. But if h is in V, the support space of μ_x,then it is unique. If $h, h' \in V$, then $h - h'$ belongs to both $V \& W$. So $h - h' = 0$ or $h = h'$.

Lemma 4.4. *If there is an element h in the set A with $\mu_x(A) = \langle h, x \rangle$, then every $a \in A$ satisfies the condition $0 \le \langle a, x \rangle \le 2\langle h, x \rangle$.*

Proof. Since $\langle h, x \rangle = \mu_x(A) = \sup\{\langle a - b, x \rangle : a, b \in A\}$, for any $a \in A$, we have $\langle h - a, x \rangle \le \sup\{\langle a - b, x \rangle : a, b \in A\} = \langle h, x \rangle$ (by taking h in place of a and a in place of b in the definition of $\mu_x(A)$). This can be simplified to get $0 \le \langle a, x \rangle$. For $a \in A$, $\langle a - h, x \rangle \le \sup\{\langle a - b, x \rangle : a, b \in A\} = \langle h, x \rangle$ (by taking h in place of b in the definition of $\mu_x(A)$). This can be simplified to $\langle a, x \rangle \le 2\langle h, x \rangle$ for all $a \in A$.

The next theorem says that the fuzzy measure of a closed convex subset of the Hilbert space resembles the length of a closed interval in R as a difference between the end points.

Theorem 4.5. *If A is a closed convex subset of H, then there exists a pair of vectors $h, k \in A$ satisfying $\mu_x(A) = |\langle h - k, x \rangle|$.*

Proof. Without loss of generality it is assumed that the vector $h \in H$ satisfying $\mu_x(A) = \langle h, x \rangle$ is in A. (Otherwise, making use of the translation invariance of the $VGFM$, a suitable translate of A may be considered). Any $a \in A$ satisfies the inequality $0 \le \langle a, x \rangle \le 2\langle h, x \rangle$. We consider the two parts of this inequality separately.

- Part I: $0 \le \langle a, x \rangle$ for all $a \in A$.

Then $\langle h - a, x \rangle \le \langle h, x \rangle = \mu_x(A)$ for all $a \in A$. Since $\mu_x(A)$ is defined as a supremum, there exists an element $a_1 \in A$ such that $\langle h, x \rangle - 1 < \langle h - a_1, x \rangle \le \langle h, x \rangle$. Having chosen $a_1, a_2, \cdots, a_n \in A$, choose $a_{n+1} \in A$ such that $\langle h, x \rangle - \frac{1}{n+1} < \langle h - a_{n+1}, x \rangle \le \langle h, x \rangle$ and $\|a_{n+1} - a_n\| < \frac{1}{2^n}$. (This is possible since A is a convex subset of the Hilbert space H and using the definition of supremum). Now, (a_n) is a Cauchy sequence which converges to some $k \in A$. By continuity of the non-zero linear functional $y \mapsto \langle y, x \rangle$ it follows that $\langle h - k, x \rangle = \langle h, x \rangle = \mu_x(A)$.

- Part II: $\langle a, x \rangle \le 2\langle h, x \rangle$ for all $a \in A$.

Then $\langle a - h, x \rangle \le \langle h, x \rangle$ for all $a \in A$. As in Part I, a sequence (a_n) can be chosen from A that converges to a vector $k \in A$ satisfying $\langle k - h, x \rangle = \langle h, x \rangle = \mu_x(A)$.

Hence there are vectors $h, k \in A$ satisfying $\mu_x(A) = |\langle h - k, x \rangle|$ and thus completing the proof.

Remark 4.6. The pair of vectors $h, k \in A$ is not necessarily unique. As an example, consider $H = R^3$, $x = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ and $A = \{(x, y, 0) : 0 \le x \le 1, 0 \le y \le 2\}$. The pair $h = (0,0,0)$, $k = (1.2.0)$ as well as the pair $h = (0,2,0)$, $k = (1.0.0)$ give $\mu_x(A) = |\langle h - k, x \rangle| = 3/\sqrt{2}$.

References

[1] George Bachman and Lawrence Narici, Functional analysis, Academic press, New York, (1966).

[2] F. S. De blasi and N.v. Zhivkov, Properties of typical bounded lcosed convex sets in Hilbert space, Abstract and applied analysis, *Hindawi Publishing Corporation,* 4 (2005), 423-436.

[3] Manju Cherian and K. Sudheer,Vector generated fuzzy measures on Hilbert spaces, *Bulletin of Kerala Mathematics Association,* Vol.5, No.2 December (2009), 63-67.

[4] Mila Stojokovic and Zoran Stojokovic, Integral with respect to fuzzy measure in finite dimensional Banach spaces, *Novi Sad J. Math.,* Vol.37, No.1 (2007), 163-170.

[5] Endre Pap, Regular Null additive monotone set functions, *Univ. u Novom Sadu Zb. Rad. period. Mat. Fak. Ser. Mat.,* 25 (2) (1995), 93-101.

[6] Endre Pap, The continuity of the null additive fuzzy measures, *Novi Sad J. Math.,* Vol.27, No.2 (1997), 37-47.

[7] Walter Rudin, Real and complex analysis, T M H Publishing company Ltd. New Delhi, (1966).

[8] Zhenyuan Wang and George J. Klir, Fuzzy measure theory, Plenum press, New York and London, (1992).