40

1M1N24051 (Pages : 2) Reg. No:........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

First Semester M.Sc Mathematics Degree Examination, November 2024

MMT1C01 – Algebra – I

(2022 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A

Answer all questions. Each carries 1 weightage

1. Verify whether $\phi(x, y) = (x + 1, y + 2)$ is an isometry of the plane.
2. Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?
3. State Burnside's Formula.
4. Find isomorphic refinements of $\{0\} < 10\mathbb{Z} < \mathbb{Z}$ and $\{0\} < 25\mathbb{Z} < \mathbb{Z}$.
5. Find the reduced form and the inverse of the reduced form of the word $a_2^3 a_2^{-1} a_3 a_1^2 a_1^{-1}$.
6. A group of order 255 must have either ——— or ——Sylow 3-subgroups.
7. Give a presentation of $\mathbb{Z}_4$ involving three generators.
8. Find the product of $1 + 2i + 2j$ and $1 - 2i - 3k$ in the ring of quaternions.

(8 × 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each carries 2 weightage

Unit I

9. Find the order of the factor group $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_2}{\langle (2, 1) \rangle}$.
10. Prove that a factor group of a cyclic group is cyclic.
11. Find both the centre and the commutator subgroup of $\mathbb{Z}_3 \times S_3$.

Unit II

12. Prove that every group of order 81 is solvable.

13. Let G be an abelian group of order pq, where p and q are primes and $p \neq q$. Show that G is simple.

14. Prove that every group is a homomorphic image of a free group.

Unit III

15. Let $\phi_i : \mathbb{Q}[x] \to \mathbb{C}$ be the evaluation homomorphism with $\phi_i(x) = i$. Find the kernel of ϕ_i.

16. Show that the multplicative group of all non zero elements of a finite field is cyclic.

17. Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and let N be the subring of all functions g such that $g(3) = 0$. Is N an ideal in F?. Why or why not?

($6 \times 2 = 12$ weightage)

Part C

Answer any two questions. Each carries 5 weightage

18. Let X be a G−set, $x \in$ X. Prove the following.

    (a) G_x is a subgroup of G.

    (b) $|G_x| = (G : G_x)$.

    (c) if $|G|$ is finite then $|G_x|$ is a divisor of $|G|$.

19. (a) Let H be a sugroup of G. Prove that the left coset multiplication is well defined by the equation $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.

    (b) State and prove Second Sylow Theorem.

20. (a) Let H be subgroup of G and let N be a normal subgroup of G. Show that $(HN)/N$ is isomorphic to $H/(H \cap N)$.

    (b) Prove that the group S_3 is solvable.

21. (a) Let F be a field and $f(x) \in F[x]$ be of degree 2 or 3. Show that $f(x)$ is irreducible if and only if $f(x)$ has no zero in F.

    (b) State Eisenstein criterion for irreducibilty of a polynomial.

    (c) Show that the polynomial $x^5 + 6x^3 + 4x + 10$ is irreducible in $Q[x]$.

($2 \times 5 = 10$ weightage)

41

1M1N24052 (Pages : 3) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

First Semester M.Sc Mathematics Degree Examination, November 2024

MMT1C02 – Linear Algebra

(2022 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A: Answer all questions. Each carries 1 weightage

1. Is the set $\{1, 1+x, (1+x)^2, (1+x)^3\}$ will form a basis for the space of all polynomials of degree less than or equal to 3? Justify your answer.
2. Let V *and* W be finite dimensional vector spaces over the field F. Prove that V *and* W are isomorphic if and only if $\dim V = \dim W$.
3. Let T be the unique linear operator on $\mathbb{C}^3$ for which $T\varepsilon_1 = (1,0,i)$, $T\varepsilon_2=(0,1,1)$, $T\varepsilon_3=(i,1,0)$ is T invertible?
4. Let W_1 *and* W_2 be subspaces of a finite dimensional vector space V. Prove that $(W_1+W_2)^0 = W_1^0 \cap W_2^0$.
5. If A and B be $n \times n$ matrices over a field F, then prove that AB and BA have precisely the same characteristic values of F.
6. Prove that an orthogonal set of non zero vectors in an inner product space is linearly independent
7. If T is a linear operator on R^2, represented in the standard ordered basis by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then prove that the only subspaces of R^2 which are invariant under T are zero subspace and R^2
8. If S is any subset of an inner product space V, then Prove that the orthogonal compliment of S, $S^{\perp}$ is a subspace of V.

$(8\times 1 = 8\ weightage)$

Part B: Answer any two questions from each unit. Each carries 2 weightage

Unit I

9. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose V is finite dimensional then prove that $rank\ (T) + nullity\ (T) = dim\ V$.

10. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If T is invertible then prove that the inverse function T^{-1} is a linear transformation. Also give one example.

11. Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over the field F then prove that the space $L(V,W)$ is finite dimensional and has dimension mn.

Unit II

12. If W is the subspace of $\mathbb{R}^5$ which is spanned by the vectors α_1= (2, -2, 3, 4, -1) , α_2 = (-1, 1, 2, 5, 2), α_3 = (0, 0, -1, -2, 3) and α_4= (1, -1, 2, 3, 0) then find the annihilator of W.

13. If W is a k-dimensional subspace of an n - dimensiopal vector space V, then prove that W is the intersection of $(n-k)$ hyperspaces in V

14. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W, then prove that there exist a unique linear transformation T' from W^* in to V^* such that $(T' g)(\alpha)$=g(T α) for every g in W^* and α in V.

Unit III

15. Apply Gram-Schmidt process to the vectors β_1=(2, 0, -2), β_2=(1, 0, 1), β_3=(0, 3, 3) to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

16. Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then Prove that E is an idempotent linear transformation of V onto W, $W^\perp$ is the null space of E and $V = W \oplus W^\perp$

17. If $K > 2$, then prove that the subspace $W_1, \ldots. W_k$ are independent if and only if

$W_j \bigcap (W_1 + \ldots + W_{j-1} + W_{j+1} + \ldots + W_k) = \{0\}$

(6 x 2 = 12 weightage)

Part C: Answer any two questions. Each carries 5 weightage

18. (a) Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j, j = 1,2,\ldots,n$ **(3 weightage)**

(b) If A is an $m \times n$ matrix with entries in the field F, then prove that *row rank (A) = column rank (A).* **(2 weightage)**

19. Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then prove that $f(T) = 0$.

20. (a) find an invertible matrix P such that $P^{-1}AP$ is a diagonalizable matrix where

$$A=\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$ **(2 weightage)**

(b) Let T be a linear operator on the finite dimensional space V. Let $c_1, c_2, \ldots c_k$ be the distinct characteristic vector of T and let W_i be the space of characteristic vector associated with the characteristic value c_i. If $W= W_1+W_2+\ldots+W_k$ then prove that $\dim W=\dim W_1+\dim W_2+\ldots+\dim W_k$ **(3 weightage)**

21. (a) let T be a linear operator on a finite dimensional space V. If T is diagonalizable and $c_1, c_2, \ldots c_k$ be the distinct characteristic vector of T then prove that there exist linear operators $E_1, E_2, \ldots E_k$ on V such that

(i) $E_iE_j = 0 \quad i \neq j$

(ii) E_i is a projection

(iii) The range of E_i is the characteristic space for T associated with c_i. **(3 weightage)**

(b) Prove that every finite dimensional inner product space has an orthonormal basis **(2 weightage)**

(2 x 5 =10 weightage)

42

1M1N24053 (Pages : 2) Reg. No:........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

First Semester M.Sc Mathematics Degree Examination, November 2024

MMT1C03 – Real Analysis – I

(2022 Admission onwards)

Time : 3 Hours. Maximum : 30 Weights.

Part A *Answer ALL questions. Each question carries 1 weight.*

1. Define a countable set. Prove that the union of two countable sets is countable.

2. Define a metric. Give example of a metric on $\mathbb{R}$ and verify the conditions.

3. **Prove/Disprove** $(A \cup B)^\circ = A^\circ \cup B^\circ$, where E° denotes the interior of the set E.

4. Let $A \subseteq \mathbb{R}$ be a compact set and $f : A \longrightarrow \mathbb{R}$ be continuous. Prove that $f(A)$ is compact.

5. Give example of a continuous real valued function for which the derivatives upto order two only exist at the origin and establish it.

6. State the L Hospital rule. Will the rule work for a complex function ? Explain.

7. Define the unit step function $I(x)$. If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$, then prove that $\int_a^b f \, d\alpha = f(s)$.

8. State the theorem of change of limit and derivative. Explain the related terms. **$8 \times 1 = 8$ Weights.**

Part B *Answer any TWO questions from each unit. Each question carries 2 weights.*

UNIT I

9. Prove that a neighbourhood is a convex set.

10. Define a connected set. Give one example.

 Prove that image of a connected set under a continuous function is a connected set.

11. Prove that a monotone function can not have discontinuity of the second kind.

 Also prove that the set of discontinuities is countable.

UNIT II

12. State and prove the generlised mean value theorem.

13. State and prove the Taylor's theorem.

14. Prove that a continuous function $f \in \mathscr{R}(\alpha)$ over $[a,b]$ where α is an increasing function on $[a,b]$.

UNIT III

15. Let f maps $[a,b]$ into $\mathbb{R}^k$ and $\mathbf{f} \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a,b]$. Prove that $|\mathbf{f}| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b \mathbf{f}\, d\alpha\right| \leq \int_a^b |\mathbf{f}|\, d\alpha$.

16. Give example of a sequence of functions converging uniformly in the corresponding domain. Explain the steps.

17. Let $\{f_n\}$ be a pointwise bounded sequence of complex functions on a countable set E. Prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ suc that $\{f_{n_k}(x)\}$ converges for every $x \in E$. **$6 \times 2 = 12$ Weights.**

Part C *Answer any TWO questions. Each question carries 5 weights.*

18. a) Define an algebraic number.

Give examples (one each) of an algebraic number and a number that is not algebraic.

b) Prove that the set of algebraic numbers is countable.

19. a) Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ for $n = 1, 2, \cdots$, then prove that $\cap_{n=1}^{\infty} I_n$ is non-empty.

b) Prove that a k-cell in $\mathbb{R}^k$ is compact.

20. a) State and prove the fundamental theorem of calculus.

b) If γ' is continuous on $[a,b]$, then prove that γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|dt$.

21. a) Estblish the existence of a real continuous function on $\mathbb{R}$ that is nowhere differentiabble.

b) Define equicontinuity of a family of functions. Give example of one such family.

$2 \times 5 = 10$ Weights.

43

1M1N24054 (Pages : 2) Reg. No:........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

First Semester M.Sc Mathematics Degree Examination, November 2024

MMT1C04 – Discrete Mathematics

(2022 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A (Short Answer Questions)(Answerall questions. Each question has weightage1)

1. Prove that every graph has an even number of vertices of odd degree.
2. Prove that κ $(K_{3,3}) = 3$.
3. Prove that every connected graph contains a spanning tree.
4. Is every partial order is a total order? Justify your answer.
5. Let $(X,+,.,')$ be a Boolean algebra. Prove that $x.x = x$ for all $x \in X$.
6. Tabulate the values of the Boolean function $f(x_1, x_2, x_3) = x_1x_2 + x_2'x_3$.
7. Find a dfa that accepts all strings on $\{0, 1\}$ except those containing the substring 001.
8. Define a language L and L*, the star closure of L. Give an example.

(8x1=8 weightage)

Part B

(*Answer any two question from each unit.Each question carries weightage 2*)

Unit 1

9. State and prove a necessary and sufficient condition for a graph to be bipartite.
10. If G is a simple graph , then prove that $\kappa(G) \le \kappa'(G) \le \delta(G)$.
11. Derive the Euler's formula for a connected plane graph.

Unit 11

12. Let $X = R \cup \{*\}$ where * is some point not on the real line. Define $\le$ on X as $\{(x, y) \in R \times R: x \le y \text{ in the usual order }\} \cup \{(*,*)\}$. Prove that $\le$ is a partial order on X.
13. State and prove Stone representation theorem for finite Boolean algebra.
14. Prove that every Boolean functions of n variables $x_1, x_2, \ldots x_n$ can be uniquely expressed as a sum of terms of the form $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ where each $x_i^{\varepsilon_i}$ is x_i or x_i'.

Unit 111

15. Find a dfa that accepts the set of all strings on $\sum = \{a, b\}$ starting with prefix ab.

16. Find a grammar that generate the language $L = \{a^n b^n : n \geq 0\}$.

17. Show that the language $L=\{awa:w \in \{a,b\}^*\}$ is regular.

(6x2=12 weightage)

Part C

Answer any two from the following four questions. Each question has weightage 4

18. (a) If G is a 3-regular graph, then prove that $\kappa(G) = \kappa'(G)$

    (b)State and prove Whitneys theorem on 2-connected graphs.

19. (a) Prove that $\delta(G) \leq 5$, for any simple planar graph G.

    (b)Let G be a connected graph. Prove that the following statements are equivalent

    (i) G is Eulerian.

    (ii) The degree of each vertex of G is an even positive integer.

    (iii) G is an edge –disjoint union of cycles.

20. (a) Let $(X,+,.,')$ be a Boolean algebra. Prove that the relation $\leq$ on X defined by $x \leq y$ if $x.y' = 0$ is a lattice and 0 and 1 are the minimum and maximum elements of this lattice.

    (b) Prove that the set of all symmetric Boolean functions of n Boolean variables $x_1, x_2, \ldots, x_n$ is a sub algebra of the Boolean algebra of all Boolean functions of these variables. Also prove it is isomorphic to the power set Boolean algebra of the set $\{0,1,\ldots,n\}$.

21. (a) Write the following Boolean function in the disjunctive normal form

    $F(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(x_1' + x_2 + x_3')(x_1 + x_2' + x_3')(x_1' + x_2' + x_3')(x_1 + x_2 + x_3')$.

    (b) Let L be the language accepted by a non deterministic finite accepter $M_N = (Q_N, \sum, \delta_N, q_0, F_N)$. Then prove that there exist a dfa $M_D = (Q_D, \sum, \delta_D, \{q_0\}, F_D\}$ such that $L=L(M_D)$.

(2x5=10 weightage)

1M1N24055 (Pages : 2) 44 Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

First Semester M.Sc Mathematics Degree Examination, November 2024

MMT1C05 – Number Theory

(2022 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A

Answer *all* questions. Each carry 1 weightage

1. Find all integers n such that $\varphi(n) = n/2$.
2. If $a|b$, prove that $\varphi(a)|\varphi(b)$.
3. Give example for two arithmetic functions which are multiplicative, but not completely multiplicative.
4. Prove that $\psi(x) = \sum_{m\le\log_2 x} \vartheta\left(x^{1/m}\right)$.
5. Prove that $\sum_{p\le x} \frac{\log p}{p} = \log x + O(1),\ x \ge 1$.
6. Prove or disprove: "Legendre's symbol $(n|p)$ is a completely multiplicative function of n."
7. Explain affine transformation.
8. Determine those odd primes p for which $(-3|p) = 1$.

(8 × 1 = 8 weightage)

Part B

Answer any *two* questions from each unit. Each carry 2 weightage

Unit 1

9. Prove that $\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) = -\sum_{d|n} \mu(d) \log d$, for $n \ge 1$.
10. State and prove the Selberg identity.
11. Prove that $\sum_{n\le x} \Lambda(n) \left[\frac{x}{n}\right] = log[x]!,\ x \ge 1$.

Unit 2

12. State and prove Abel's identity.
13. Prove that $\sum_{n\le x} \psi\left(\frac{x}{n}\right) = xlog\, x - x + O(logx), x \ge 1$.
14. Prove that $p_n < 12\left(nlog\, n + n\log\frac{12}{e}\right)$.

Unit 3

15. Prove that $(p|q)(q|p) = (-1)^{\frac{(p-1)(q-1)}{4}}$, for odd primes p and q.

16. Prove that $(n|p) \equiv n^{\frac{p-1}{2}} (mod\ p), \forall n$, where p is an odd prime.

17. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \in M_2(\mathbb{Z}/26\mathbb{Z})$.

(6 × 2 = 12 weightage)

Part C

Answer any *two* questions. Each carry 5 weightage

18. If a and b are positive real numbers such that $ab = x$, prove that

$$\sum_{\substack{q,d \\ qd \le x}} f(d)g(q) = \sum_{n \le a} f(n)G\left(\frac{x}{n}\right) + \sum_{n \le b} g(n)\, F\left(\frac{x}{n}\right) - F(a)G(b).$$

19. Prove that (i). $\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$.

(ii) $\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t}\, dt$.

20. State and prove Shapiro's Tauberian theorem.

21. In a long string of ciphertext, which was encrypted by means of an affine map on single-letter message units in the 26-letter alphabet, you observe that the most frequently occurring letters are "Y" and "V", in that order. Assuming that those cipher text message units are the encryption of "E" and "T", respectively, read the message "QAOOYQQEVHEQV".