45

2M3N24089 (Pages : 2) Reg. No:......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2024

MMT3C11 – Multivariable Calculus & Geometry

(2022 Admission onwards)

Time: 3hours Max. Weightage :30

Part A

Answer all questions. Each question carries one weightage.

1. Prove that if $A \in L(R^n, R^m)$, then $||A|| < \infty$ and A is a uniformly continuous mapping of R^n Into R^m.
2. Define dimension of a vector space. Show that $dim R^n = n$.
3. Prove that any reparametrization of a regular curve is regular.
4. Verify whether $\sigma(u, v) = (u, v^2, v^3); u, v \in R$ a regular surface patch or not.
5. Show that $x^2 + y^2 + z^4 = 1$ is a smooth surfaces.
6. Calculate the first fundamental form of the surface $\sigma(u, v) = (\cos u\ , \sin u\ , v)$. What kind of surface is this?.
7. Show that the Weingarten map changes sign when the orientation of the surface changes.
8. Compute the second fundamental form of the elliptic paraboloid $\sigma(u, v) = (u, v, u^2 + v^2)$.

(8x1= 8 weightage)

Part B

Answer any two questions from each unit. Each carries two weightage.

Unit 1

9. Let Ω be the set of all invertible linear operator on R^n, show that

   (a) If $A \in \Omega, B \in L(R^n)$, and $||B - A||.||A^{-1}|| < 1$, then $B \in \Omega$.

   (b) Ω is an open subset of $L(R^n)$, and the mapping $A \to A^{-1}$ is continuous on Ω.

10. Show that, if f maps an open set $E \subset R^n$ into R^m, then $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.
11. If $[A]$ and $[B]$ are n by n matrices, then show that $\det([B][A]) = \det[B] \det[A]$.

Unit 2

12. Show that a parametrized curve has a unit-speed reparametrization if and only if it is regular.
13. Let γ be a unit-speed curve in R^3 with constant curvature and zero torsion. Prove that γ is a parametrization of (part of) a circle.
14. Suppose that two smooth surfaces S and $\tilde{S}$ are diffeomorphic and that S is orientable. Prove that $\tilde{S}$ is orientable.

Unit 3

15. What is meant by an oriented surface? Show that Mobius band is not orientable.
16. Calculate the Gaussian curvature of $\sigma(u, v) = (f(u)cosv, f(u)sinv, g(u))$ where $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$.
17. Prove that the area of a surface patch is unchanged by reparametrization.

(6x2=12 weightage)

Part C

Answer any two questions. Each carries 5 weightage

18. State and prove inverse function theorem.
19. (a) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that $dimX \leq r$.

    (b) Suppose X is a vector space, and $dimX = n$. Prove that a set E of n vectors in X if and only if E is independent.

20. Let $\gamma(t)$ be a regular curve in R^3 with nowhere vanishing curvature. Prove that its torsion is given by $\tau = \frac{(\dot{\gamma} X \ddot{\gamma}).\dddot{\gamma}}{||\dot{\gamma} X \ddot{\gamma}||^2}$, where X indicate the vector product and the dot denotes d/dt.
21. Let S and $\tilde{S}$ be surfaces and let $f: S \to \tilde{S}$ be a smooth map. Then, prove that f is a local diffeomorphism if and only if, for all $p \in S$, the linear map

    $D_p f: T_p S \to T_{f(p)}\tilde{S}$ is invertible.

(2x5=10 weightage)

45

2M3N24090 (Pages : 2) Reg. No:......................

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FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2024

MMT3C12 – Complex Analysis

(2022 Admission onwards)

Time: 3 hours Max. Weightage :30

Part A

Answer all questions. Each question has weightage 1.

1. Find the point in unit sphere S in R^3 corresponding to the point $1 + i$ in C.
2. Evaluate the cross ratio $(i - 1, \infty, 1 + i, 0)$.
3. Prove that a branch of Logarithm function is analytic and its derivative is $\frac{1}{z}$.
4. Evaluate the integral $\int_{\gamma} \frac{dz}{z^2 + a}$ where $\gamma(t) = 2e^{it}$, $0 \leq t \leq 2\pi$
5. Prove or disprove: A non constant entire function is not bounded.
6. Prove that if f has an isolated singularity at a point $z = a$ and $\lim_{z \to a}(z - a)f(z) = 0$, then $z = a$ is a removable singularity
7. State Rouche's theorem.
8. Let $f(z) = [(1 - z^4)]^{-1}$. Determine the poles of f.

($8\times 1 = 8$ Weightage)

Part B

Answer six questions choosing two from each unit. Each question has weightage 2.

Unit 1

9. Let G be either the whole plane C or some open disk. If $U: G \to R$ is a harmonic function, then show that U has a harmonic conjugate.
10. Show that a Mobius transformation S is the composition of translations, dilations and the inversions.
11. Prove that the cross ratio of 4 distinct points in the extended complex plane is a real number if and only if they all lie on a circle.

Unit2

12. Let G be a connected open set and let f:G→C be an analytic function. Prove that if { z∈ G: f(z)=0} has a limit point in G, then there is a point a in G such that $f^{(n)}(a) = 0$ for each n ≥ 0.
13. Let G be a region and $f: G \to C$ be a non constant analytic function on G. Prove that $f(U)$ is open for all open set $U$of G.
14. Define the winding number of a closed rectifiable curve γ around a point not on {γ}. Prove that it is always an integer.

Unit 3

15. Prove that if f has an essential singularity at z = a , then f(z) comes arbitrarily close to every complex number as z approaches a.
16. Let f be a meromorphic function on a region G. Show that neither the poles nor the zeros of f have a limit point.
17. Let G be a region in C and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a}|f(z)| \le$ M for all a in the boundary ∂_∞G in C_∞. Prove that $|f(z)| \le$ M for all z in G.

(6× 2 = 12 Weightage)

Part C
Answer any two questions.
Each question has a weightage 5.

18. Let γ:[a b]→C be piecewise smooth.
    (a) Show that γ is of bounded variation and the total variation $v(\gamma) = \int_a^b |\gamma'(t)|dt$
    (b) Suppose that f: [a b]→C is continuous. Then prove that $\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)dt$.
19. (a) Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a given power series and $\frac{1}{R} = \lim sup|a_n|^{\frac{1}{n}}$. Prove the following.
    (i) If $|z-a| < R$, then the series converges absolutely
    (ii) If $|z-a| > R$, the series diverges.
    (b) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2+1}$
20. State and prove Laurent series development.
21. Let G be an open set and let f: G→C be a differentiable function. Prove that f is analytic on G.

(2× 5 = 10 Weightage)

46

2M3N24091 (Pages : 2) Reg. No:.......................

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FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2024

MMT3C13 – Functional Analysis

(2022 Admission onwards)

Time: 3 hours Max. Weightage :30

Part A

Answer* all *questions. Each question carries* 1 *weightage.

1. Define Metric space and Normed space. Give an example for each.
2. Show by an example that there are divergent sequences (x_n) in l^p such that the sequence $(x_n(j))$ converges in the scalar field for each $j = 1, 2, \ldots$...
3. Prove by an example that a bounded set need not be totally bounded.
4. Let X and Y be normed spaces and $F: X \to Y$ be linear. If F is bounded on $\overline{U}(0, r)$ for some $r > 0$ then prove that $\|F(x)\| \leq \alpha\|x\|$ for all $x \in X$ and some $\alpha > 0$.
5. Let X be a normed space over $\mathbb{K}$, and f be non-zero linear functional on X. If E is an open subset of X, then show that $f(E)$ is an open subset of $\mathbb{K}$.
6. Show that the linear space C_{00} cannot be a Banach space in any norm.
7. Let X and Y be normed spaces and $F: X \to Y$ be linear. Prove that F is continuous if and only if $g \circ F$ is continuous $\forall g \in Y'$.
8. If a bijective map F is closed, then prove that F^{-1} is also closed.

(8x1 = 8 weightage)

Part B

Answer any* two *questions from each unit. Each question carries* 2 *weightages.

UNIT- I

9. Sate and prove Holder's inequality for measurable functions.
10. If $m(E) < \infty$ and $1 \leq p < \infty$, then show that the set of all bounded continuous function on E is dense in $L^p(E)$.
11. Let E be a convex subset of a normed space X. Show that E° and $\bar{E}$ are also convex.

UNIT- II

12. Show that a linear functional f on a normed space X is continuous iff $Z(f)$ is closed in X.

13. Let X be a normed space over $\mathbb{K}$, $f \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then Prove that $U(a,r) \cap Z(f) = \emptyset$ if and only if $\|f\| \leq \frac{1}{r}$.

14. Prove that a Banach space cannot have a denumerable (Hamel) basis.

UNIT- III

15. Let X and Y be normed spaces, $F: X \to Y$ is continuous and $G: X \to Y$ is closed. Show that $F + G: X \to Y$ is closed.

16. If X is a normed space and $P: X \to X$ is a projection, then show that P is a closed map if and only if the subspace $R(P)$ and $Z(P)$ are closed in X.

17. State and prove Bounded inverse Theorem.

(6x2 = 12 weightages)

Part C

Answer any two questions. Each question carries 5 weightages.

18. (a) For $1 \leq p < \infty$, prove that the metric space l^p is separable.

    (b) Let Y be a closed subspace of a normed space X. For $x + Y \in X/Y$, define $|||x + Y||| = \inf\{\|x + y\| : y \in Y\}$. Show that $|||.|||$ is a norm on X/Y.

19. (a) Let X and Y be normed spaces with X finite dimensional. Then prove that every bijective linear map from X to Y is a homeomorphism.

    (b) State and prove the Hahn-Banach separation theorem.

20. (a) Show that a normed space X is a Banach space iff every absolute summable series of elements in X is summable in X.

    (b) Let X and Y be normed spaces and $X \neq \{0\}$. Then show that $BL(X, Y)$ is a Banach space in the operator norm if and only if Y is a Banach space.

21. State and prove closed graph theorem.

(2x5 = 10 weightages)

2M3N24092 (Pages : 3) Reg. No:......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS). KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2024

MMT3C14 – PDE & Integral Equations

(2022 Admission onwards)

Time: 3 hours Max. Weightage :30

Part A

Answer all questions. Each question carries 1 weightage

1. Solve the equation $-yu_x + xu_y = u$ with the initial condition $u(x,\ 0) = f(x)$.
2. Find the quadrants in which the following equation is hyperbolic.

$$(1-\sqrt{xy}\,)u_{xx} + 2u_{xy} + (1+\sqrt{xy}\,)u_{yy} = 0$$

3. If $u(x,t)$ is the solution of the Cauchy problem

$$u_{tt} - u_{xx} = 0\,;0 < x < \infty, t > 0,$$
$$u(0,t) = t^2\,;\ t > 0,$$
$$u(x,0) = x^2\,;0 \le x < \infty,$$
$$u_t(x,0) = 6x\,;\ 0 \le x < \infty,$$

evaluate the value of $u(1,4)$

4. Define Dirichlet problem, Neumann problem and Robin problem.
5. Show that the Dirichlet problem in a bounded domain:

$$\Delta u = f(x,y);\quad (x,y) \in D$$
$$u(x,y) = g(x,y);\quad (x,y) \in \partial D$$

has at most one solution in $C^2(D) \cap C(\bar{D})$.

6. Find the integral equation which is equivalent to the IVP $y' - y = 0,\ y(0) = 1$.
7. Determine $p(x)$ and $q(x)$ in such a way that the equation $x^2y'' - 2xy' + 2y = 0$ is equivalent to the equation $\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = 0$.
8. Suppose that the solution of the integral equation $f(x) = x + \int_0^1 x\xi\, f(\xi)\, d\xi$, has the form $f(x) = ax$. Find the value of a.

(8 x 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each question carries 2 weightage

Unit I

9. Suppose that the equation $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$, where $a, b, \ldots, f, g$ are given functions of x and y, is elliptic in a planar domain D. Assume further that a, b, c are real analytic functions in D. Show that there exists a coordinate system (ξ, η) in which the equation has the canonical form

$$w_{\xi\xi} + w_{\eta\eta} + l_1[w] = G(\xi, \eta),$$

where l_1 is a first-order linear differential operator, and G is a function which depends on the given PDE.

10. Solve the Cauchy problem

$$u_{tt} - u_{xx} = t^7 \,;\ -\infty < x < \infty, \qquad t > 0$$
$$u(x,0) = 2x, + \sin x, \quad u_t(x,0) = 0\,;\ -\infty < x < \infty.$$

11. Show that the Cauchy problem $u_x + u_y = 1$; $u(x,x) = x$ has infinitely many solutions. Find at least two of them.

Unit II

12. Use the method of separation of variables to solve the Heat equation with homogeneous boundary conditions.

13. Solve the problem

$$u_{tt} - u_{xx} = \cos 2\pi x . \cos 2\pi t\ ;\ 0 < x < 1, \qquad t > 0,$$
$$u_x(0,t) = u_x(1,t) = 0\,;\ t \geq 0$$
$$u(x,0) = \cos^2 \pi x\,;\ 0 \leq x \leq 1$$
$$u_t(x,0) = 2\cos 2\pi x\,;\ 0 \leq x \leq 1$$

14. Solve the Laplace equation $\Delta u = 0$ in the square $0 < x, y < \pi$, subject to the boundary condition $u(x,0) = u(x,\pi) = 1$, $u(0,y) = u(\pi,y) = 0$.

Unit III

15. Prove that the equation $y(x) = F(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+\xi)\ y(\xi)\,d\xi$

possesses no solution when $F(x) = x$, but that it possesses infinitely many solutions when $F(x) = 1$. Determine all such solutions.

16. Show that the characteristic numbers of a Fredholm integral equation

$y(x) = \lambda \int_a^b K(x,\xi)\, y(\xi)\, d\xi$, with a real symmetric kernel $K(x,\xi)$, are all real.

17. Write a note on Neumann series.

(6 x 2 = 12 weightage)

Part C

Answer any two questions. Each question carries 5 weightage

18. (a) Derive d'Alembert's formula for the Cauchy problem for the one-dimensional homogenous wave equation.

(b) State and prove the existence and uniqueness theorem for the Cauchy problem of first order Quasilinear equations.

19. (a) Apply the method of separation of variables to solve the problem:

$$u_{tt} - c^2 u_{xx} = 0\,; \quad 0 < x < L, \qquad t > 0,$$
$$u_x(0,t) = u_x(L,t) = 0\,; \quad t \geq 0,$$
$$u(x,0) = f(x); \quad 0 \leq x \leq L,$$
$$u_t(x,0) = g(x)\,; \; 0 \leq x \leq L.$$

(b) State and prove The weak maximum principle and The strong maximum principle.

20. (a) Use the iterative method to solve the equation $y(x) = 1 + \lambda\int_0^1 (1 - 3x\,\xi)\, y(\xi)\, d\xi$.

(b) Determine the eigen values and eigen functions of the integral equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+\xi)\ y(\xi)\, d\xi$$

21. Discuss the method of Lagrange to solve first order Quasilinear PDE and use it to solve the Cauchy problem $-yu_x + xu_y = 0;\quad u(x,0) = \sin x,\ x > 0$

(2 x 5 = 10 weightage)

48

2M3N24093 (Pages : 2) Reg. No:.......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2024

MMT3E03 – Measure & Integration

(2022 Admission onwards)

Time: 3 hours Max. Weightage :30

Part A

Answer all questions. Each question carries 1 weightage.

1. Let X be a measurable space and f be a complex measurable function on X. Then prove that there exist a complex measurable function α such that $|\alpha| = 1$ and $f = \alpha|f|$.
2. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then prove that $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\mu(A_1)$ is finite.
3. Let $\{E_k\}$ be a sequence of measurable sets in X, such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$.Then prove that almost all x in X lie in at most finitely many of the sets E_k.
4. Define lower semi continuous function. Give an example of a lower semicontinuous which is not upper semicontinuous.
5. Let X be an uncountable set and $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is countable. Prove that $\mathfrak{M}$ is a σ-algebra in X.
6. Suppose μ, λ_1 and λ_2 are measures on a σ-algebra $\mathfrak{M}$ and μ is positive. If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then prove that $\lambda_1 \perp \lambda_2$.
7. Let f be $(\mathcal{S} \times \mathcal{T})$-measurable on $X \times Y$. Then prove that for each $x \in X$, f_x is a $\mathcal{T}$-measurable function on Y.
8. Give an example to show that σ-finiteness is necessary in the hypothesis of Fubini theorem.

Part B

Answer two questions from each unit. Each question carries 2 weightage.

Unit I

9. State and prove Lebesgue's dominated convergence theorem.
10. Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then prove that $\alpha f + \beta g \in L^1(\mu)$ and $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$.
11. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then prove that there is a σ-algebra $\mathfrak{M}^*$ which is the μ-completion of the σ-algebra $\mathfrak{M}$.

Unit II

12. Let X be locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then prove that λ is regular.
13. If μ is a complex measure on X, then prove that $|\mu|(X) < \infty$.
14. State and prove Hahn Decomposition theorem.

Unit III

15. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be measure spaces If $E \in \mathcal{S} \times \mathcal{T}$, then prove that $E_x \in \mathcal{T}$ and $E^y \in \mathcal{S}$ for every $x \in X$ and $y \in Y$
16. State and prove Fubini theorem.
17. Let m_k denote Lebesgue measure on $\mathbb{R}^k$. If $k = r + s, r \geq 1, s \geq 1$, then prove that m_k is the completion of the product measure $m_r \times m_s$.

Part C

Answer any two questions. Each question carries 5 weightage.

18. (i). Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then prove that $\alpha f + \beta g \in L^1(\mu)$ and $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$.

    (ii) If $f \in L^1(\mu)$, then prove that $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.
19. State and prove the Vitali-Caratheodory Theorem.
20. Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$. Then prove that there is a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that $\lambda = \lambda_a + \lambda_s$, $\lambda_a \ll \mu, \lambda_s \perp \mu$. Further prove that there is a unique $h \in L^1(\mu)$ such that $\lambda_a = \int_E h d\mu$ for every set $E \in \mathfrak{M}$.
21. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be measure spaces. Then prove that $\mathcal{S} \times \mathcal{T}$ is the smallest monotone class which contains all elementary sets.