1M3N23089 (Pages : 2) Reg. No:.......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2023

MMT3C11 – Multivariable Calculus & Geometry

(2022 Admission onwards)

Time: 3hours

Max. Weightage :30

Part A

Answer all questions, Each question has weightage 1.

1. Suppose X is a vector space, and $dim\ R^n = n$. Prove that a set E of n vectors in X spans X if and only if E is independent.
2. Let $A \in L(R^n, R^m)$. Prove that $\|A\| < \infty$
3. Show that $det[A]_1 = -\det[A]$, if $[A]_1$ is an nxn matrices obtained from $[A]$ by interchanging two columns.
4. Define a parametrized curve. Find a parametrization of the level curve $y^2 - x^2 = 1$.
5. Define arc length of a curve γ. Calculate the arc length of the catenary $\gamma(t) = (t, cos\ t)$ Starting at the point $(0,1)$.
6. Prove that any reparametrization of a regular curve is regular.
7. Calculate the first fundamental form of the surface: $\sigma(u, v) = (\cosh\ u, \sinh\ u, v)$.
8. Compute the second fundamental form of the elliptic paraboloid
$$\sigma(u, v) = (u, v, u^2 + v^2).$$

(8 x 1= 8 weightage)

Part B

Answer six questions choosing two from each unit

Each question has weightage 2

UNIT I

9. Let r be a positive integer. If a vector space X is spanned by a set of r vectors then prove that $dimX \leq r$.
10. Let f maps an open set $E \subset R^n$ into R^m and f is differentiable at a point $x \in E$. Prove that $(D_j f_i)(x)$) exist, and $f^1(x)e_j = \sum_{i=1}^{m}(D_j f_i)(x)$)u_i. $(1 \leq j \leq n)$
11. Let $[A]$ and $[B]$ are n by n matrices. Prove that $\det([B][A]) = \det[B]\det[A]$.

UNIT II

12. Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

13. Let γ is a regular closed curve of period T. Prove that its unit speed reparametrization of γ is always closed.

14. Let γ be a unit-speed curve in R^3 with constant curvature and zero torsion. Prove that γ is a parametrization of (part of) a circle.

UNIT III

15. Prove that the area of a surface patch is unchanged by reparametrization.

16. Calculate the Gaussian curvature of $\sigma(u,v) = \big(f(u)cosv, f(u)sinv, g(u)\big)$ where $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$.

17. Calculate the principal curvatures of the helicoids $\sigma(u,v) = (vcos\,u, vsin\,u, \lambda u)$.

(6 x 2 = 12 weightage)

Part C

Answer two questions

Each question has weightage 5.

18. State and prove inverse function theorem.

19. (a) Let the distance between A and B in $L(R^n, R^m)$ be defined as $||A - B||$.Prove that $L(R^n, R^m)$ is a metric space.

    (b) Prove that if X is a complete metric space and φ is a contraction from X into X, then φ has a unique fixed point in X.

20. Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in R^3 with the same curvature $\kappa(s) > 0$ the same torsion $\tau(s)$ for all s. Then, there is a direct isometry M of R^3 such that $\tilde{\gamma}$(s)=M $(\gamma(s))$ for all s. Further, if k and t are smooth functions with $k > 0$ everywhere, there is a unit-speed curve in R^3 whose curvature is k and whose torsion is t.

21. Let S be a connected surface of which every point is an umbilic. Then, prove that S is an open subset of a plane or a sphere.

(2 x 5 = 10 weightage)

32

1M3N23090 (Pages : 2) Reg. No:......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2023

MMT3C12 – Complex Analysis

(2022 Admission onwards)

Time: 3 hours Max. Weightage :30

Part A

Answer all questions. Each question has weightage 1.

1. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{z^n}{n!}$
2. Let T be a Mobius transformation. Show that $T(0) = \infty$ and $T(\infty) = 0$ if and only if $T(z) = \frac{a}{z}$ for some complex number a.
3. Show that $f(z) = x^2 + y^2$ has a derivative only at the origin where $x = Re(z)$ and $y = Im(z)$.
4. Evaluate the integral $\int_{\gamma} \frac{(2z+1)}{z^2+z+1} dz$ where γ is the circle $|z| = 2$.
5. Show that a bounded entire function is a constant.
6. Obtain the Laurent series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ for $1<|z|<2$.
7. Give an example of a meromorphic function that has infinitely many simple poles.
8. State Schwarz's lemma.

(8× 1 = 8 Weightage)

Part B

Answer six questions choosing two from each unit. Each question has weightage 2.

Unit 1

9. Discuss the stereographic projection
10. Prove that the cross ratio of 4 distinct points in the extended complex plane is a real number if and only if they all lie on a circle.
11. If G is open and connected and f: G→C is differentiable with $f'(z) = 0$ ∀z in G, prove that f is a constant.

Unit2

12. Are the zeroes of an analytic function isolated? Justify your answer.

13. State and prove Maximum Modulus theorem.

14. Let $\gamma: [0,1] \to C$ be a closed rectifiable curve and $a \notin \gamma$. Show that $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer.

Unit 3

15. Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

16. Prove that if f has an essential singularity at $z = a$, then $f(z)$ comes arbitrarily close to every complex number as z approaches a.

17. Let f be a meromorphic function on a region G. Show that neither the poles nor the zeros of f have a limit point.

(6× 2 = 12 Weightage)

Part C

Answer any two questions.
Each question has a weightage 5.

18. Let G and Ω be open subsets of C and $f: G \to C$ and $g: \Omega \to C$ be functions such that $f(G) \subseteq \Omega$.
    (a) Let f and g be analytic on G and Ω respectively. Show that $g \circ f$ is analytic and finds its derivative.
    (b) If f is continuous on G, g is differentiable, $g(f(z)) = z$ for all $z \in G$ and $g'(z) \neq 0$, then prove that f is differentiable and $f'(z) = \frac{1}{g'(f(z))}$

19. Find an analytic function $f: G \to \mathbb{C}$ where G={ z: Re z>0} such that $f(G) = \{z: |z| < 1\}$.

20. Let G be open in C and let γ be a rectifiable path in G with initial and end points α and β respectively. Then prove that $\int f = F(\beta) - F(\alpha)$, where $f: G \to C$ is a continuous function with a primitive $F: G \to C$.

21. (a)State and prove Residue theorem.
    (b) Find a pole a and residue at a of the function $f(z) = z^2(1 + z^4)^{-1}$.

(2× 5 = 10 Weightage)

33

1M3N23091

(Pages : 2)

Reg. No:......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2023

MMT3C13 – Functional Analysis

(2022 Admission onwards)

Time: 3 hours

Max. Weightage :30

Part A

Answer all questions. Each question carries 1 weightage.

1. Let d be the discrete matrix and d' be the usual metric on X, then show that d is stronger than d'.
2. Prove that every totally bounded set is bounded.
3. If a sequence (x_n) in the metric space l^p, $1 \leq p \leq \infty$, converges to x in l^p, then prove that $(x_n(j))$for each $j = 1, 2, \ldots..$ converges to $(x(j))$ in $\mathbb{K}$.
4. Let X and Y be normed spaces. If X is finite dimensional, then show that every linear map from $X \to Y$ is continuous.
5. Let X be a linear space over $\mathbb{C}$. Regarding X as a linear space over $\mathbb{R}$, consider a real linear functional $u: X \to \mathbb{R}$. Define $f(x) = u(x) - iu(ix), x \in X$. Prove that f is a complex-linear functional on X.
6. Let X be a normed space over $\mathbb{K}$, and f be non-zero linear functional on X. If E is an open subset of X, then show that $f(E)$ is an open subset of $\mathbb{K}$.
7. Let X and Y be normed spaces and $F: X \to Y$ be linear. Prove that F is continuous if and only if $g \circ F$ is continuous $\forall g \in Y'$.
8. Let X and Y be normed spaces. If Z is a closed subspace of X, then show that the quotient map Q from X to X/Z is continuous.

(8 X 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each question carries 2 weightages.

UNIT- I

9. Show that the metric space l^p is complete for $1 \leq p < \infty$.
10. If $m(E) < \infty$ and $1 \leq p < \infty$, then show that the set of all bounded continuous function on E is dense in $L^p(E)$.
11. If $n \geq 2$, then show that $\mathbb{K}^n$ with the norm $\| \|_2$ is strictly convex.

UNIT- II

12. Let X and Y be normed spaces with X finite dimensional. Then prove that every bijective linear map from X to Y is a homeomorphism.

13. State and prove the Hahn-Banach separation theorem.

14. Show that a normed space X is a Banach space iff every absolute summable series of elements in X is summable in X.

UNIT- III

15. State and prove the uniform boundedness principle for Banach spaces.

16. Let X be a normed space and E be a subset of X. Then prove that E is bounded in X if and only if $f(E)$ is bounded in $\mathbb{K}$ for every $f \in X'$.

17. Let X and Y be normed spaces and $F: X \to Y$ be linear. Then prove that F is an open map if there exists some $\gamma > 0$ such that every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

(6 X 2 = 12 weightages)

Part C

Answer any two questions. Each question carries 5weightages.

18. (a) Sate and Minkowski's inequality for sequences.

    (b) State and prove Riesz lemma.

19. Let X be a normed space. Prove that for every subspace Y of X and every $g \in Y'$, there is a unique Hahn-Banach extension of g to X if and only if X' is strictly convex.

20. (a)Let X be normed space and Y be a closed subspace of X. Show that X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in the induced norm and the quotient norm, respectively.

    (b)Prove that a Banach space cannot have a denumerable (Hamel) basis.

21. (a) State and prove the open mapping theorem.

    (b) State and prove Bounded inverse theorem.

(2 X 5 = 10 weightages)

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(Pages : 3)

Reg. No:......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2023

MMT3C14 – PDE & Integral Equations

(2022 Admission onwards)

Time: 3 hours

Max. Weightage :30

Part A

Answer all questions. Each question carries 1 weightage

1. Show that the Cauchy problem $u_x = c_0 u,\ u(x,0) = 2e^{c_0 x}$, where c_0 is a constant, has infinitely many solutions.
2. Consider the equation $u_{xx} + 2u_{xy} + [1 - q(y)]u_{yy} = 0$, where

$$q(y) = \begin{cases} -1 & ;\ y < -1 \\ 0 & ;\ |y| \le 1 \\ 1 & ;\ y > 1 \end{cases}$$

   Find the domains where the equation is parabolic, elliptic and hyperbolic.
3. Define Dirichlet problem, Neumann problem and Robin problem.
4. If $u(x,t)$ is the solution of the Cauchy problem

$$u_{tt} - u_{xx} = 0; \quad 0 < x < \infty, \qquad t > 0,$$
$$u(x,0) = \cos\left(\frac{\pi}{2}x\right); \quad 0 \le x < \infty,$$
$$u_t(x,0) = 0; \quad 0 \le x < \infty,$$
$$u_x(0,t) = 0; \quad t > 0,$$

   evaluate $u(2,2)$.
5. Derive a necessary condition for the existence of a solution to the Neumann problem.
6. Reduce the Volterra integral equation $y(x) = x - \cos x + \int_0^x (x-\xi)y(\xi)\,d\xi$ to equivalent initial value problem.
7. Show that the kernel $K(x,\ \xi) = \sin x . \cos \xi$ has no characteristic numbers associated with $(0,\ 2\pi)$.
8. Check whether the kernel defined by $k(x,\xi) = \cos(x+\xi)$ is separable or not.

(8 x 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each question carries 2 weightage

Unit I

9. Explain the characteristics method to solve the initial value problem associated with first order Quasilinear PDEs and use it to solve $u_x = 1,\ u(0,y) = g(y)$; where $g(y)$ is a function of y.

10. Reduce the equation $u_{xx} + 6u_{xy} - 16u_{yy} = 0$ into its canonical form. Hence find its general solution.

11 Solve the Eikonal equation $p^2 + q^2 = {n_0}^2,\ u(x, 2x) = 1$.

Unit II

12. Solve the problem

$$u_t = u_{xx}\ ; \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u(\pi,t) = 0\ ; \ t \geq 0$$
$$u(x,0) = \begin{cases} x\ ;\ 0 \leq x \leq \pi/2 \\ \pi - x\ ;\ \pi/2 < x \leq \pi \end{cases}$$

13. Use the method of separation of variables to solve the Wave equation with homogeneous boundary conditions.

14. Using the energy method prove uniqueness of the problem

$$u_t - ku_{xx} = F(x,t)\ ;\ 0 < x < L, t > 0,$$
$$u(0,t) = a(t), \quad u(L,t) = b(t)\ ;\ t \geq 0$$
$$u(x,0) = f(x)\ ;\ 0 \leq x \leq L$$

Unit III

15. Determine the eigen values and eigen functions of the integral equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \sin(x + \xi)\ y(\xi)\ d\xi$$

16. Show that the characteristic functions of the homogeneous Fredholm integral equation $y(x) = \lambda \int_a^b K(x,\xi)\, y(\xi)\, d\xi$, with the symmetric kernel $K(x,\xi)$, corresponding to distinct characteristic numbers are orthogonal over (a,b).

17. Derive the formula $\underbrace{\int_a^x \cdots \int_a^x}_{n\ times} f(x)\, dx \ldots dx = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi$

(6 x 2 = 12 weightage)

Part C

Answer any two questions. Each question carries 5 weightage

18. (a) Show that the Cauchy problem $(y+u)u_x + yu_y = x - y; \quad u(x,1) = 1 + x$ has a unique solution. Also find the solution u as a function of x and y.

(b) Prove that the equation $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ is parabolic and find its canonical form. Also find the general solution on the half plane $x > 0$.

19. (a) Solve the Laplace equation in the square $0 < x, y < \pi$, subject to the Dirichlet condition $u(x,0) = 1984,\ u(x,\pi) = u(0,y) = u(\pi,y) = 0$.

(b) State and prove The mean value principle.

20. (a) Discuss the iterative method for solving Fredholm integral equations of second kind.

(b) Use Green's function to transform the boundary value problem

$y'' + y = x\,; y(0) = 0,\ y'(1) = 1$ to a Fredholm Integral equation.

21. (a) Use the Lagrange method to find a function $u(x,y)$ that solves the Cauchy problem

$$uu_x + u_y = 1; \quad u(3x,0) = -x, \qquad -\infty < x < \infty$$

(b) Show that the curve $\{(3x, 2, 4-3x) : -\infty < x < \infty\}$ is contained in the solution surface $u(x,\ y)$.

(c) Solve $uu_x + u_y = 1; \quad u(3x,2) = 4 - 3x,\ -\infty < x < \infty$

(2 x 5 = 10 weightage)

35

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(Pages : 2)

Reg. No:.......................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Third Semester M.Sc Mathematics Degree Examination, November 2023

MMT3E03 – Measure & Integration

(2022 Admission onwards)

Time: 3 hours

Max. Weightage :30

Part A

Answer all questions. Each question carries 1 weightage.

1. Let E be a measurable set in a measurable space X and $\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E \end{cases}$. Then prove that χ_E is a measurable function on X.
2. Prove that if f is a real function on a measurable space X such that $\{x | f(x) \geq r\}$ is measurable set for every rational r, then prove that f is measurable.
3. If $f \in L^1(\mu)$, then prove that $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.
4. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then prove that $\lambda_1 + \lambda_2 \perp \mu$
5. State Lebesgue Radon Nykodym theorem.
6. Define Jordan decomposition of a real measure μ on a sigma algebra $\mathfrak{M}$. State the minimum property of the Jordan decomposition.
7. Suppose that $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. If $E \in \mathcal{S} \times \mathcal{T}$, then prove that $E_x \in \mathcal{T}$
8. Prove that Fubini theorem fails if the measures are not σ-finite.

Part B

Answer two questions from each unit. Each question carries 2 weightage.

Unit I

9. State and prove Lebesgue's monotone convergence theorem.
10. Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e on X such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. Then prove that the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for almost all x, $f \in L^1(\mu)$ and $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.
11. State and prove Urysohn's Lemma.

Unit II

12. Prove that every set of positive measure has non measurable subset.
13. Suppose $f \in L^1(\mu)$, f is real valued and $\epsilon > 0$. Prove that there exists functions u and v such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below and $\int_X (v - u)d\mu < \epsilon$.
14. Prove that the total variation $|\mu|$ of a complex measure μ on $\mathfrak{M}$ is a positive measure on $\mathfrak{M}$.

Unit III

15. Suppose $1 \leq p < \infty$, μ is a σ-finite positive measure on X, $\mu(X) < \infty$ and Φ is a bounded linear functional on $L^p(\mu)$. Then prove that there is a function $g \in L^1(\mu)$ such that $\Phi(f) = \int_X fgd\mu$.
16. Prove that the class of elementary set is an algebra.
17. State and prove Fubini theorem.

Part C

Answer any two questions. Each question carries 5 weightage.

18. Prove that every measure μ on a σ-algebra can be completed.
19. State and prove Lusin's theorem.
20. Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$. Then prove that there is then a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that $\lambda = \lambda_a + \lambda_s$, $\lambda_a \ll \mu, \lambda_s \perp \mu$. Further prove that there is a unique $h \in L^1(\mu)$ such that $\lambda_a = \int_E hd\mu$ fo every set $E \in \mathfrak{M}$.
21. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be σ-finite measure spaces. Suppose $\varphi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q^y)$ for every $x \in X$ and $y \in Y$, then prove that φ is $\mathcal{S}$-measurable, ψ is $\mathcal{T}$-measurable and $\int_X \varphi d\mu = \int_Y \psi d\lambda$.