1M2A23083

(Pages : 2) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Second Semester M.Sc Mathematics Degree Examination, April 2023

MMT2C06 – Algebra II

(2022 Admission onwards)

Time: 3 hours

Max. Weightage : 30

Part A

Answer all questions. Each question carries 1 weightage

1. Prove that a finite extension is an algebraic extension.
2. Define algebraically closed field. Give example.
3. Determine all isomorphisms of $Q(\sqrt{2})$into$\bar{Q}$
4. Let K be the splitting field of $x^4 + 1$ over Q. Show that $G(K/Q)$ is isomorphic to klein-4 group
5. Find all automorphisms of the field Z_p where p is a prime
6. Let F be a field of characteristic zero and let $a \in F$. Let K be the splitting field of $x^n - a$ over F. Suppose F contain all the n^{th} root of unity. Prove that $G(K/F)$ is abelian.
7. Define maximal ideal of a ring R and give one example.
8. Show that the field R of reals is not algebraically closed. Also find the algebraic closure of R

Part B

Answer any two questions from each unit. Each question carries 2 weightage

Unit I

9. Let H be a subgroup of the Galois group $G(K/Q)$.
   Show that $K_H = \{a \epsilon K : \sigma(a) = a\}$ is a subfield of K.
10. If F is a field then show that every ideal in F [X] is principal
11. Find all prime ideals of Z_n

12. Prove that a maximal ideal in a commutative ring with unity is a prime ideal.

13. Is the regular 150-gon constructable ? Give reason.

14. Prove that a field is perfect if every extension is a separable extension.

Unit III

15. Let α be a real cube root of 2. Verify that $Q(\alpha)$ is not a splitting field. also find the splitting field of $\{x^3 - 2, x^3 - 3\}$ over Q.

16. Find $\Phi_8(x)$ over Z_3.

17. Give an example for an extension E of $Q\ (2^{\frac{1}{3}})$ such that $[E: Q] = \{E: Q\} = |G(E/Q)|$.

Part C

Answer any two questions from each unit. Each question carries 5 weightage

18. Let E be an extension of a field F and let $\alpha \epsilon E$. Prove that

    (a) $\varphi_\alpha: F[X] \to E$ defined by $f(x) \to F(\alpha)$ for $f(x) \in F[X]$ is a homomorphism

    (b) If α is algebraic over F, then $Ker\ \varphi_\alpha \neq \{0\}$.

    (c) If α is transcendental over F then φ_α is one-one.

19. (a)State and prove Conjugation isomorphism theorem.

    (b)Prove that Complex zeros of polynomials with real coefficients occur in conjugate pairs

20. (a) Prove that $x^2 - 3$ is irreducible over $Q(2^{\frac{1}{3}})$

    (b)Let E be a finite extension of a field F and K be a finite extension of a field E. Prove that K is a finite extension of field F and $[K: F\} = [K: E][E: F]$

21. (a) Show that Doubling the cube is impossible.

    (b) Describe the $\Phi_8(x)$ over Q. Show that $\Phi_8(x) = x^4 + 1$.

39

1M2A23084 (Pages : 3) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Second Semester M.Sc Mathematics Degree Examination, April 2023

MMT2C07 – Real Analysis – II

(2022 Admission onwards)

Time: 3 hours

Max. Weightage : 30

Part-A

Answer all questions. Each question carries 1 weightage

1) State and prove excision property of Lebesgue outer measure.
2) If A and B are measurable subsets of $\mathbb{R}$, prove that $A \cup B$ is measurable.
3) If f is measurable, show that for any extended real number c, the set $\{x | \ f(x) = c\}$ is measurable
4) Let ϕ and ψ be simple functions defined on a set of finite measure E. Prove that

$$\int_E (\phi + \psi) = \int_E \phi + \int_E \psi$$

5) State and prove the integral comparison Test.
6) Let $\{f_k\}_1^n$ be a finite family of functions, each of which is integrable over E. Show that $\{f_k\}_1^n$ is uniformly integrable and tight over E.
7) Let $f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & if\ 0 < x \le 1 \\ 0 & if\ x = 0 \end{cases}$. Is f of bounded variation on $[0,1]$?
8) If the function f is Lipschitz on a closed, bounded interval $[a, b]$, prove that it is absolutely continuous on $[a, b]$.

Part-B

Answer any two questions from each unit. Each question carries 2 weightage

Unit - I

9) Prove that every interval is measurable.
10) Prove that any set of real numbers with positive outer measure contains a subset which is not measurable.
11) Let the function f be defined on a measurable set E. Prove that f is measurable if and only if for each open set O, $f^{-1}(O)$ is measurable.

Unit - II

12) Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E and $\{f_n\} \to f$ uniformly on E. Prove that

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

13) State and prove the Lebesgue Dominated Convergence theorem.

14) Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Prove that $\{f_n\} \to f$ in measure on E.

Unit - III

15) Let f be an increasing function on the closed, bounded interval $[a, b]$. Prove that for each a $\alpha > 0$, $m^*\{x \in (a,b) | \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$ and $m^*\{\{x \in (a,b) | \overline{D}f(x) = \infty\} = 0$

16) Prove that a function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

17) Let E be a measurable set and $1 \leq p \leq \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Prove that $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \int_E \|f_n\|_p = \int_E \|f\|_p$

Part-C

Answer any two questions. Each question carries 5 weightage

18)

a) If $\{A_k\}_1^\infty$ is an ascending collection of measurable sets, then prove that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$$

b) Let $\{f_n\}_1^\infty$ be a sequence of measurable functions on E that converges pointwise $a.e.$ on E to the function f. Prove that f is measurable.

19)

a) State and prove the bounded convergence theorem.

b) Let the functions f and g be integrable over E. Prove that for any α and β, the function $\alpha f + \beta g$ is integrable over E and $\int_E(\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$

20)

a) Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, prove that f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

b) Prove that a function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$

21) Let the function f be continuous on the closed, bounded interval [a, b]. Prove that f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{Diff_h\, f\}_{0<h<1}$ is uniformly integrable over $[a, b]$.

1M2A23085 (Pages : 2) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Second Semester M.Sc Mathematics Degree Examination, April 2023

MMT2C08 – Topology

(2022 Admission onwards)

Time: 3 hours Max. Weightage : 30

Part A

Answer* all *questions. Each question carries* 1 *weightage.

1. Define the Sierpinski space. Is it a topology not induced by any metric.
2. Define Base for topology. Write a sub-base for the set of real numbers under usual topology.
3. Prove that a second countable space always contains a countable dense subset.
4. Prove that a subset of a topological space is open if and only if it is a neighborhood of each of its points.
5. Define weak topology determined by a family of functions.
6. Show that components are closed sets.
7. Give an example of a topological space that is T_1 but not T_2.
8. Define standard base and standard sub-base for the product topology.

(8x1 = 8 weightage)

Part B

Answer any* two *questions from each unit. Each question carries* 2 *weightages.

UNIT- I

9. Le t X be a set and $\mathfrak{D}$ a family of subsets of X. Then prove that there exists a unique topology $\mathcal{T}$ on X, such that it is the smallest topology on X containing $\mathfrak{D}$.
10. Prove that if a space is second countable then every open cover of it has a countable subcover.
11. Prove that composition of continuous functions is continuous.

UNIT- II

12. Define divisible topological property. Prove that the property of being a discrete space is divisible.
13. What is mean by a Lebesgue number. Prove that every continuous image of a compact space is compact.
14. Let $\mathcal{C}$ be a collection of connected subsets of a space X such that no two members of $\mathcal{C}$ are mutually separated. Then prove that $\bigcup_{C\in\mathcal{C}} C$ is also connected.

UNIT- III

15. Prove that the intersection of any family of boxes is a box and the intersection of finite number of large boxes is a large box.
16. If the product is non-empty, then prove that each co-ordinate space is embeddable in it.
17. Let $X = \prod_{i\in I} X_i$, each X_i being a topological space. Suppose $\{x_n\}$ is a sequence in X and that $x \in X$. Then prove that $\{x_n\}$ converges to x iff for each $i \in I$, the sequence $\{\pi_i(x_n)\}$ converges to $\pi_i(x)$ in X_i.

(6x2 = 12 weightages)

Part C

Answer any two questions. Each question carries 5 weightages.

18. (a) Let $(X,\mathcal{T})$ be a topological space and $\mathcal{S}$ a family of subsets of X. Then prove that $\mathcal{S}$ is a sub-base for $\mathcal{T}$ if and only if $\mathcal{S}$ generates $\mathcal{T}$.
    (b) Prove that metrisability is a hereditary property.
19. (a) Prove that a subset A of a space X is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \emptyset$.
    (b) For a subset A of a space X, prove that $\bar{A} = A \cup A'$.
20. (a) Prove that every continuous real valued function on a compact space is bounded and attains its extrema.
    (b) Show that every second countable space is first countable. Is the converse true? Justify your answer.
21. (a) Prove that in a Hausdorff space, limits of sequences are unique.
    (b) Show that regularity is a hereditary property.

(2x5 = 10 weightages)

1M2A23086 (Pages : 2) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Second Semester M.Sc Mathematics Degree Examination, April 2023

MMT2C09 – ODE and Calculus of Variations

(2022 Admission onwards)

Time: 3 hours

Max. Weightage : 30

Part A

(Answer all questions. Each question carries 1 weightage)

1. Find a power series solution of the equation $y' = y$.
2. Verify that $p_n(-1) = (-1)^n$, where $p_n(x)$ is the n^{th} degree Legendre polynomial.
3. If $y(x)$ be a nontrivial solution of equation $y'' + q(x)y = 0$ on a closed interval $[a, b]$, $then$ $y(x)$ has atmost a finite number of zeros in this interval.
4. Describe the phase portrait of the system: $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2$.
5. Using Picard's method, find the solution of $y' = y$ with the initial condition $y(0) = 1$.
6. Prove that $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$.
7. Determine whether the function $2x^2 - 3xy + 3y^2$ is positive definite, negative definite or neither.
8. Show that $f(x, y) = x^2|y|$ satisfies a Lipschitz condition on the rectangle $|x| \leq 1$and $|y| \leq 1$.

(8 X 1 = 8 weightage)

Part B

(Answer any two questions from each unit. Each question carries 2 weightage)

Unit I

9. Derive the orthogonality property of the Legendre Polynomial.
10. Find the general solution of $y'' + xy' + y = 0$.
11. Find two independent Frobenius series solutions of the equation $2xy'' + (3 - x)y' - y = 0$.

Unit II

12. If $W(t)$ is the Wronskian of the two solutions of the homogeneous linear system

$$\left\{\begin{array}{l}\frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y\end{array}\right\}$$, then $W(t)$ is either identically zero or nowhere zero on $[a,b]$.

13. State Bessel Expansion theorem and compute the Bessel series of the function $f(x) = 1$ for the interval $0 \le x \le 1$ interms of the function $J_0(\lambda_n x)$, where λ_n's are the positive zeros of $J_0(x)$.

14. Determine the nature and stability properties of the critical point (0,0) for the System $\frac{dx}{dt} = 4x - 3y$; $\frac{dy}{dt} = 8x - 6y$.

Unit III

15. State and Prove Sturm comparison theorem.

16. Formulate the problem of finding the curve of quickest descent.

17. Find the exact solution of the initial value problem $y' = 2x(1+y)$, $y(0) = 0$. Starting with $y_0(x) = 0$, calculate $y_1(x), y_2(x), y_3(x), y_4(x)$ and compare these results with the exact solution.

(6 X 2 = 12 weightage)

Part C

(Answer any two questions. Each question carries 5 weightage)

18. Derive the Rodrigue's formula for Legendre polynomials $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

19. If there exists a LiapunovfuctionE(x,y) for the autonomous system, then the critical point (0,0) is stable. Furthermore, if this function has the additional property that the function $\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$ is negative definite, then the critical point (0,0) is asymptotically stable.

20. State and Prove Picard's theorem.

21. For the system $\frac{dx}{dt} = -x, \frac{dy}{dt} = 2x^2y^2$

 (i) Find the critical points.

 (ii) Find the differential equation of the paths.

 (iii)Solve this equation to find the paths

(2 X 5 = 10 weightage)

42

fM2A23087 (Pages : 3) Reg. No:.........................

Name: ....

FAROOK COLLEGE (AUTONOMOUS), KOZHIKODE

Second Semester M.Sc Mathematics Degree Examination, April 2023

MMT2C10 – Operations Research

(2022 Admission onwards)

Time: 3 hours

Max. Weightage : 30

Part A (Short Answer Questions)

(Answer all questions. Each question has weightage 1)

1. Prove that the sum of two convex functions is a convex function.
2. What is meant by degeneracy in a linear programming problem?
3. Solve graphically the following linear programming problem:

   Minimize $20x_1 + 50x_2$

   Subject to $x_1 + 2x_2 \geq 10, 3x_1 + 4x_2 \leq 24, x_1 \geq 0, x_2 \geq 0$
4. Prove that the dual of the dual is the primal.
5. What type of problems can be solved by the dual simplex method? Explain.
6. Formulate the transportation problem as a linear programming problem.
7. Define a chain in a graph. When does a chain become a cycle?
8. Describe the rectangular game as a linear programming problem.

$(8 \times 1 = 8\ weightage)$

Part B

(Answer any *two* questions from each unit. Each question carries weightage 2)

Unit I

9. Let $X \in E_n$ and $f(X) = X'AX$ be a quadratic form. If $f(X)$is positive semidefinite, then show that $f(X)$ is a convex function.
10. Prove that a basic feasible solution of the LP problem is a vertex of the convex set of feasible solutions.
11. Solve the following LP problem by simplex method:

    Maximize $x_1 + x_2 + 3x_3$

    Subject to

$$3x_1 + 2x_2 + x_3 \leq 3$$
$$2x_1 + x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

Unit II

12. Prove that the optimal value of $f(X)$ of the primal, if it exists, is equal to the optimal value of $\emptyset(Y)$ of the dual.

13. Prove that the transportation problem has a triangular basis.

14. Solve the following transportation problem for minimum cost starting with the degenerate solution $x_{12} = 30, x_{21} = 40, x_{32} = 20, x_{43} = 60$.

	D_1	D_2	D_3	
O_1	4	5	2	30
O_2	4	1	3	40
O_3	3	6	2	20
O_4	2	3	7	60
	40	50	60	

Unit III

15. Describe the algorithm for finding the minimum path between two vertices in a graph when all arc lengths are non-negative.

16. Solve the following LP problem by the cutting plane method:

Minimize $4x_1 + 5x_2$

Subject to

$$3x_1 + x_2 \geq 2$$
$$x_1 + 4x_2 \geq 5$$
$$3x_1 + 2x_2 \geq 7$$

x_1, x_2 non-negative integers.

17. Let $f(X,Y)$ be such that both $\max_X \min_Y f(X,Y)$ and $\min_Y \max_X f(X,Y)$ exist. Then prove that $\max_X \min_Y f(X,Y) \leq \min_Y \max_X f(X,Y)$.

$(6 \times 2 = 12$ weightage$)$

Part C

(Answer any *two* from the following four questions. Each question carries weightage 5)

18. Solve the following LP problem by big M method:

Minimize $2x_1 - 3x_2 + 6x_3$

Subject to

$$3x_1 - 4x_2 - 6x_3 \leq 2$$
$$2x_1 + x_2 + 2x_3 \geq 11$$
$$x_1 + 3x_2 - 2x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

19. (a) Prove that if the k^{th} constraint of the primal is an equality, then the dual variable y_k is unrestricted in sign.

(b) Solve the following LP problem by dual simplex method:

Minimize $2x_1 + 3x_2$

Subject to $2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1, x_2 \geq 0$

20. Solve graphically the LP problem: Maximize $f = 4x_1 + 8x_2$ subject to

$x_1 + 2x_2 \geq 20,\ 2x_1 + 2x_2 \leq 100,\ x_1 - 3x_2 \leq 0,\ 4x_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0.$

Also analyse graphically how the optimal solution is modified when the following changes are introduced in the problem, (one at a tifne);

(*i*) objective function is replaced by $8x_1 + 4x_2$

(*ii*) a new constraint $2x_1 + x_2 \geq 10$ is introduced.

21. (a) State and prove the fundamental theorem of rectangular games.

(b) Solve graphically the game whose payoff matrix is $\begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix}$

$(2 \times 5 = 10$ weightage)